Math 558 Fall 2014 — Prof. J. Rauch

Homework 8 — Due Tuesday November 4

Only six questions because problems 4 is long.

1. Exercise 6.5 of Spectral Theorem Handout.

2. Exercise 6.6 of Spectral Theorem Handout.

3. Exercose 6.7 of the Spectral Theorem Handout.

4. For the two spring system of 136/7 show that if the mass whose location is x_1 is subject to friction, that is it has an extra force $-ax_1'$ with $a > 0$, then all the eigenvalues of the corresponding 4×4 matrix have strictly negative real part.

Hint. It may be possible to do this by brute force. Or you can reason as follows.

• Introduce a natural energy and derive the dissipation law for solutions,

$$\frac{dE}{dt} \;=\; -\,a\,|x_1'|^2\,. \tag{1}$$

The kinetic energy is clear. The potential energy comes from the expansion or compression of the springs. Add the kinetic and potential energies. It is conserved for $a = 0$. Equation (1) implies that the energy of all solutions is non increasing.

• Use decrease of energy to prove that all eigenvalues lie in $\{\mathrm{Re}\, z \le 0\}$.

• If there were a purely imaginary eigenvalue there would be a nonzero solution $U(t) := e^{i\omega t}v$. Show that the real and imaginary parts of $U(t)$ are solutions whose energy is periodic. Show that the energy of those solutions must be constant.

• Use dissipation to show that a solution with constant energy has constant x_1.

• Show that a solution with constant x_1 is identically equal to zero.

Discussion. This is a non trivial physically reasonable example of asymptotic stability.

5. a. Draw the phase line for

$$x' = -x(x^2 - \varepsilon^2), \qquad 0 < \varepsilon << 1\,.$$

Explain why observers not sensitive to small scales might be deceived into thinking that the equilibrium at 0 is stable while it is not.

b. In contrast, explain how the equilibrium $x = 0$ for $x' = x(x^2 - \varepsilon^2)$ is stable but to all but the most careful experimenter, appears unstable.

6. i. Find all equilibrium solutions of the system

$$x' = x^2 + y^2 - 1, \qquad y' = 2xy\,.$$

ii. Where possible, determine their stability from the linearization.