Math 558 Fall 2006 Prof. J.Rauch
Final exam December 20, 2006

Instructions.

1. Clearly explain your answers.
2. You are allowed two sides of a 3"×5" card of notes.
3. No calculators.
4. There are 5 problems and a total of 50 points.

Good Luck.

1. (13 points) The origin is an equilibrium point of each of the nonlinear systems

$$\begin{aligned} x' &= x + y + xy \\ y' &= -y + x^2 + y^2\,, \end{aligned}$$

$$\begin{aligned} x' &= x + y + x^2 - y^2 \\ y' &= y + xy\,. \end{aligned}$$

i. (4+4 points) *The origin is what type of equilibrium in each case?* For example, one of them is a saddle.

ii. (5 points) *For the saddle compute the tangent line to the stable manifold at the origin.*

2. (8 points) Consider the system

$$x' = -x + a\,y\,, \qquad y' = ax - 2y\,,$$

with a a real constant. *Show that the function*

$$L(x,y) \;=\; 2\,x^2 + y^2\,,$$

is a strict Lyapunov function on the entire plane $\mathbb{R}^2$ *if and only if* $|a| < 4/3$.

3. (6+4 points) The planar system in polar coordinates,

$$r' = r - 1 + (r-1)^2 \sin\theta\,, \qquad \theta' = 1\,,$$

has the 2π periodic orbit

$$r(t) = 1, \quad \theta(t) = t\,, \qquad x(t) = \cos t, \quad y(t) = \sin t\,.$$

The positive x-axis is a section, denoted S, defined in polar coordinates by $\theta = 0$. Since $\theta' = 1$, the first return from $(r, 0)$ occurs at $t = 2\pi$ at which time $\theta = 2\pi$.
Consider orbits beginning on S. Then $r = x$ and $\theta = 0$ at $t = 0$. For $x \approx 1$ define $r(t,x), \theta(t,x)$ to be the solution with

$$r(0,x) = x\,, \qquad \theta(0,x) = 0\,.$$

Then $r(t,1), \theta(t,1)$ is the periodic orbit. The function $x \to r(2\pi, x)$ is the first return map.

i. *Show that*

$$R(t) := \frac{\partial r(t,x)}{\partial x}\bigg|_{x=1}\,, \qquad \Theta(t) := \frac{\partial \theta(t,x)}{\partial x}\bigg|_{x=1}\,,$$

satisfy the variational (perturbation) equations

$$R' = R, \quad \Theta' = 0\,, \quad R(0) = 1\,, \quad \Theta(0) = 0\,.$$

Conclude that

$$\frac{dr(2\pi, x)}{dx}\bigg|_{x=1} = e^{2\pi}\,.$$

ii. *Use the result of* **i** *to show that the the first return of trajectories near the periodic orbit return roughly* $e^{2\pi}$ *times further away from the orbit* $r = 1$ *than they started.*

4. (8 points) Suppose that

$$X' = F(X)\,, \qquad X = (x_1, x_2)$$

is an autonomous planar system of ordinary differential equations such that

a. The origin is the only equilibrium inside the unit disk $\{|X| \le 1\}$ and is a source.

b. $F(X) \cdot X \le 0$ when $|X| = 1$ so the disk is positively invariant.

Show that for any $X \neq 0$ in the disk, the ω-limit set $\omega(X)$ is a periodic orbit.

5. (4+2+5) points) For $0 \neq \lambda$ consider the map

$$f_\lambda(x) \;=\; \lambda x(x-1)\,.$$

i. Show that f_λ has exactly two fixed points and find them.

ii Compute $f_\lambda^2(x)$ the second iterate of f_λ.

iii. For which values of $\lambda \neq 0$ does f_λ has a two cycle which is not a fixed point of f_λ. **Hint.** Each of the fixed points of f_λ is a fixed point of f_λ^2. This gives two roots and therefore two linear factors of the polynomial whose roots are the 2-cycles. You need to find out if the polynomial has any other roots.