Math 558 Fall 2006 | Prof. J.Rauch
Midterm exam | November 2, 2006

Instructions.

1. Clearly explain your answers.
2. You are allowed two sides of a 3”×5” card of notes.
3. No calculators.
4. There are four problems.

Good Luck.

1. The first three parts concern the scalar ordinary differential equation

$$p' \;=\; p^{100} - 1\,.$$

i. (4 points) Find all equilibrium points.

ii. (4 points) Draw the phase line diagram.

iii. (2 points) Determine the stability of each equilibrium point.

iv. (3+2 points) This equation is in the family of equations

$$p' \;=\; p^{100} + a\,, \qquad -\infty < a < \infty\,.$$

Determine all values of a where bifurcations occur and describe the change(s) that occur at the bifurcations. Diagrams can help here.

Solution outline. The equilibria are the solutions of $p^{100} - 1 = 0$. There are two, $p = \pm 1$. Since $p^{100} - 1$ is positive outside $[-1, 1]$ and negative in $]-1, 1[$, solutions move to the right when they are ouside $[-1, 1]$ and move to the left in the interval $]-1, 1[$.

It follows that the equilibrium $p = -1$ is a sink and $p = 1$ is a source. This can also be shown by linearization since the equilibria are hyperbolic.

The same analysis works for the family in iv. when $a < 0$ and the equilibria are at $\pm|a|^{1/2}$.

When $a = 0$ there is exactly one equilibrium, at $p = 0$. The phase line in this case is moving to the right at all points other than $p = 0$.

For $a > 0$ there are no equilibria and the flow is strictly right moving at all points.

2. This problem has four parts, **ai, aii, bi, bii** each worth 4 points. For each of the two systems

$$(\mathbf{a})\quad X' = \begin{bmatrix} -1 & 2 \\ -1/2 & -1 \end{bmatrix} X\,, \qquad (\mathbf{b})\quad X' = \begin{bmatrix} 4 & 5 \\ -2 & -3 \end{bmatrix} X\,,$$

i. Determine if the system falls into one of the following categories, sprial sink, spiral source, saddle, nonspiral source, nonspiral sink.

ii. If a spiral sink or spiral source determine the direction of rotation of the spiral and the exponential rate of growth or decay.

If a source determine the direction of fastest growth. If a (nonspiral) sink determine the direction of slowest decay.

If a saddle determine the direction of the stable line. (Definition. The *stable line* is the set of initial data with the property that the solution of the corresponding initial value problem tends to 0 as $t \to \infty$.

Solutionn outline. a. Compute

$$\det(A - \lambda I) = \lambda^2 + 2\lambda + 2\,.$$

The eigenvalues are $\lambda = -1 \pm i$. The equilibrium is a spiral sink.

The tangent at $(1, 0)$ has second component equal to $-1/2$ so the orbit crosses the x-axis moving in the **clockwise** direction.

The solutions decay like e^{-t} thanks to the real part of the eigenvalues.

b.

$$\det(A - \lambda I) = \lambda^2 + \lambda - 2 = (\lambda - 2)(\lambda + 1)\,.$$

The eigenvalues are $-1, 2$. The equilibrium is a **saddle**.

The stable line is the line of eigenvectors with eigenvalue -1. That is, $\mathbb{R}(1, -1)$.

3. (15 points) Define

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Find the general solution of $X' = A\,X$.

Solution outline. Since

$$A = \begin{bmatrix} A_{3\times 3} & 0 \\ 0 & A_{2\times 2} \end{bmatrix},$$

$$\det(A - \lambda I) = \det(A_{3\times 3} - \lambda I_{3\times 3})\ \det(A_{2\times 2} - \lambda I_{2\times 2}) = (\lambda+1)^3\big[(1-\lambda)^2 + 4\big].$$

The roots are $\lambda = -1, -1, -1, 1 \pm 2i$.

The eigenvectors with eigenvalue $1+2i$ are the nonzero multiples of $V := (0,0,0,-2i,1)$. Two linearly independent solutions are

$$e^{(1+2i)t}\,V\,, \qquad e^{(1-2i)t}\,\overline{V}\,,$$

For $\lambda = -1$ the eigenspace is

$$E_\lambda = \ker(A+I)^3 = \ker(A+I)^2 = \Big\{ v \,:\, v_4 = v_5 = 0 \Big\}.$$

This set is invariant and $(A+I)^2$ vanishes on these vectors so the solutions with values in E_λ are given by the three parameter family of solutions

$$e^{-t}\Big[I + (A+I)t\Big](a,b,c,0,0)\Big].$$

The general solution is then

$$e^{-t}\Big[I + (A+I)t\Big](a,b,c,0,0) + d\,e^{(1+2i)t}\,V + f\,e^{(1-2i)t}\,\overline{V}\,.$$

4. (5 points) Compute the linearization of the nonlinear system

$$x_1' \;=\; \cos x_1 \, \sin x_2 \,, \qquad x_2' \;=\; \pi \, x_1^2 - x_2 \,,$$

at the equilibrium point $(x_1, x_2) = (1, \pi)$.

Solution outlilne. Define

$$f_1(x_1, x_2) = \cos x_1 \, \sin x_2 \,, \qquad f_2(x_1, x_2) = \pi \, x_1^2 - x_2 \,.$$

The linearization at an equilibrium is the system $X' = A\,X$ with matrix A equal to the value of

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} ,$$

evaluated at the equilibrium point.
The matrix of partial derivative is equal to

$$\begin{bmatrix} -\sin x_1 \sin x_2 & \cos x_1 \cos x_2 \\ 2\pi x_1 & -1 \end{bmatrix} .$$

The matrix A is computed by plugging in $(x_1, x_2) = (1, \pi)$.