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Causality

Abstract Ordinary differential equations arise as models in an astounding number and variety of applications. This is not an accident. Any smooth stable system with a finite number of degrees of freedom **must** be modelled by an ordinary differential equation. This handout shows why.

1 Definitions.

Definition 1.1. *A system is called* **causal** *if for any* t_0 *the state at time* t_0 *uniquely determines the state at time* $t \geq t_0$*. It is called* **continuous** *if the state for* $t_0 \leq t \leq t_1$ *depends continuously in a suitable topology on its state on* t_0*. It is called* **smooth** *if the state for* $t_0 \leq t \leq t_1$ *depends in (an infinitely) differentiable way on the initial state.*

This definition lacks precision about the set of all states, the meaning of continuity or smoothness, and admissible values for t_1. The qualitative idea is what is important.

Definition 1.2. *A system is said to have a* **finite number of degrees of freedom** *if its state is determined by a finite number,* N*, of real measurements.*

The values of those measurements yield a set of points in $\mathbb{R}^N$ where N is the number of measurements. The states are in one to one correspondence with a subset of Euclidean space. The values of all other measurements are functions of the N determining measurements.

Definition 1.3. *If* N *is the smallest number possible, then the system is said to have* N **degrees of freedom***.*

In this case we suppose that that subset is open. This means that slightly changing the values of the N parameters determining the system yields values corresponding for a slightly perturbed system. In this case the notions of continuity and smoothness can be taken to be that of Euclidean space.

Example 1.4. *For a Newtonian particle in* $\mathbb{R}^3$ *the state is determined by the six measurement corresponding to position and velocity,* x, v*. The seventh measurement, kinetic energy,* K *is a function of those. If one takes* N *equal to 7, with the measurements* x, v, K *as detemining the state then the states correspond to six dimensional surface in* $\mathbb{R}^7$ *given by the equation*

$$K \; = \; m|v|^2/2\,.$$

The system has six degrees of freedom. For the six measurements the states correspond to all of $\mathbb{R}^6$*.*

2 Connection with ODE

Suppose that a system has N degrees of freedom and its states correspond to the open set of points $\Omega \subset \mathbb{R}^N$. Suppose that it is smooth and causal in the sense that for each $x_0 \in \Omega$ and t_0 there is a $t_1 > t_0$ and an $r > 0$ so that for $|x - x_0| < r$ there is a unique future state $\phi(t, t_0, x_0)$ defined for $t_0 \le t \le t_1$. And that ϕ is an infinitely differentiable function of its arguments.
Define

$$F(t,x) \;=\; \frac{\partial \phi}{\partial t}\bigg|_{(t,t,x)} .$$

F is the time derivative at time t of the trajectory which at time t is equal to x. The definition implies that every trajectory of the causal system satisfies,

$$x' = F(t,x)$$

showing that smooth causal systems are all described by a smooth system of ordinary differential equations.

A smooth causal system with N *degrees of freedom is always modelled by a system of* N *first order ordinary differential equations.*