Spectral Decomposition of General Matrices

Summary. *Sometimes there are not enough eigenvectors to form a basis. There is always a basis of generalized eigenvectors. This gives a spectral decomposition of general matrices. And yields the general solution of $X' = AX$. As a special case we derive the diagonalisability of symmetric matrices. Stability of $X' = AX$ is studied with an eye to the analysis of asymptotic stability by linearization.*

1 Generalized eigenspaces

Suppose that A is a linear transformation from a finite dimensional complex vector space $\mathbb{V}$ to itself and that λ is an eigenvalue of A. The nullspace [1] $\ker(A-\lambda I)$ is the linear space consisting of all eigenvectors with eigenvalue λ together with the zero vector.

Then $\ker(A-\lambda I)^2 \supset \ker(A-\lambda I)$ since if $(A-\lambda I)v = 0$ then surely $(A-\lambda I)(A-\lambda I)v = 0$. In this way for $\ell = 1, 2, \ldots$ define a nondecreasing sequence of linear subspaces,

$$K_\ell \;:=\; \ker\big(A-\lambda I\big)^\ell, \qquad \text{so} \qquad 0 \neq K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots .$$

Example 1.1 *The matrix*

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

has characteristic polynomial

$$\det(zI - A) \;=\; (z-3)^2,$$

so has only the eigenvalue $\lambda = 3$. For this eigenvalue,

$$K_1 \;=\; \ker(A-3I) \;=\; \mathbb{C}\,(1,0), \qquad K_2 \;=\; \ker(A-3I)^2 \;=\; \ker\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \;=\; \mathbb{C}^2.$$

So $K_\ell = \mathbb{C}^2$ for all $\ell \geq 2$.

Proposition 1.1 *There is an $r \leq \dim \mathbb{V}$ so that*

$$0 \;\neq\; K_1 \;\subsetneq\; K_2 \;\subsetneq\; \cdots \;\subsetneq\; K_r \;=\; K_{r+1} \;=\; K_{r+2} \;=\; \cdots\,,$$

with

$$\dim K_\ell \;\geq\; \ell \quad \text{for} \quad 1 \leq \ell \leq r\,. \tag{1.1}$$

Proof. Whenever there is strict inclusion the dimension must increase by at least one. Thus, there can be at most $\dim \mathbb{V}$ strict inclusions $K_\ell \neq K_{\ell+1}$. Therefore, there is a first $r \geq 1$ so that $K_{r+1} = K_r$.

[1] The nullspace or kernel of a linear transformation is the set of vectors X such that $BX = 0$. It is a linear subspace. To use the results of the note to compute general solutions one uses Gaussian elimination to compute nullspaces.

To show that the $K_j = K_r$ for $j \geq r$, it suffices to prove that $K_\ell = K_{\ell+1}$ implies $K_{\ell+1} = K_{\ell+2}$. If $K_\ell = K_{\ell+1}$ and $v \in K_{\ell+2}$, then $(A - \lambda I)^{\ell+1}(A - \lambda I)v = 0$ so $(A - \lambda I)v \in K_{\ell+1} = K_\ell$ so $(A - \lambda I)^{\ell}(A - \lambda I)v = 0$ proving that $v \in K_{\ell+1}$.
Since the inclusion of the K_ℓ is strict before r, the dimensions satisfy,

$$\dim K_1 \ \geq \ 1, \qquad \dim K_\ell \ \geq \ \dim K_{\ell-1} + 1, \quad \text{for} \quad 2 \leq \ell \leq r,$$

proving (1.1). ■

Definition 1.1 *The space K_r is called the* **generalized eigenspace** *associated to λ and r is its* **index**.

Exercise 1.1 *Suppose that $\mathbb{V} = \mathbb{C}^N$ and that A is a real matrix.* **i.** *Show that if λ is real, then the generalized eigenspace X satisfies $X = \overline{X}$ where the overline signifies complex conjugate.* **ii.** *If λ is not real show that $\overline{X}$ is equal to the generalized eigenspace associated to the eigenvalue $\overline{\lambda}$.*

Proposition 1.2 *If K_r is the generalized eigenspace associated to λ and its index is r, define*

$$Y \ := \ \text{range}\,(A - \lambda I)^r\,.$$

i. *K_r and Y are invariant under A, that is $A(K_r) \subset K_r$ and $A(Y) \subset Y$.*
ii. *One has the direct sum decomposition*

$$\mathbb{V} \ = \ K_r \ \oplus \ Y\,. \tag{1.2}$$

Proof. **i.** For the invariance of K, if $v \in K_r$ then $(A - \lambda I)^r v = 0$ so since A commutes with any polynomial in A,

$$(A - \lambda I)^r A v \ = \ A(A - \lambda I)^r v \ = \ A\,0 \ = \ 0,$$

proving that $Av \in K_r$.
For the invariance of Y, if $w \in Y$ then there is a u so that $w = (A - \lambda I)^r u$ then

$$Aw \ = \ A(A - \lambda I)^r u \ = \ (A - \lambda I)^r (Au) \ \in \ \text{Range}\,(A - \lambda I)^r \ := \ Y\,.$$

ii. K and Y are the kernel and range of the linear transformation $(A - \lambda I)^r$. It follows that

$$\dim K_r \ + \ \dim Y \ = \ \dim \mathbb{V}\,.$$

To prove (1.2) it therefore suffices to show that

$$K_r \cap Y = 0\,.$$

If $v \in K_r \cap Y$ then,

$$(A - \lambda I)^r v = 0, \qquad \text{and there is a } u \text{ s.t. } \ v = (A - \lambda I)^r u\,.$$

Then $(A - \lambda I)^{2r} u = 0$. That is, $u \in K_{2r} = K_r$ so, $v = (A - \lambda I)^r u = 0$. ■

This result splits A into two pieces, the part in K and the part in Y. The next result shows that that split separates the part with eigenvalue λ from the rest.

Proposition 1.3 *λ is the only eigenvalue of $A|_{K_r}$, and, λ is not an eigenvalue of $A|_Y$.*

Proof. To prove the first assertion suppose that $\widetilde{\lambda} \neq \lambda$ and $v \in K_r$ satisfies $Av = \widetilde{\lambda} v$. Then

$$(A - \lambda I)v = (\widetilde{\lambda} - \lambda)v\,.$$

Therefore

$$(A-\lambda I)^2 v = (A-\lambda I)(A-\lambda I)v = (A-\lambda I)(\widetilde{\lambda}-\lambda)v = (\widetilde{\lambda}-\lambda)(A-\lambda I)v = (\widetilde{\lambda}-\lambda)^2 v\,.$$

By induction one has for all k

$$(A-\lambda I)^k v \;=\; (\widetilde{\lambda}-\lambda)^k v\,.$$

Taking $k = r$ yields

$$(A-\lambda I)^r v \;=\; (\widetilde{\lambda}-\lambda)^r v\,.$$

Since $v \in K_r$ one has,

$$0 \;=\; (A-\lambda I)^r v \;=\; (\widetilde{\lambda}-\lambda)^r v,$$

so $v = 0$ since $(\widetilde{\lambda}-\lambda)^r \neq 0$.

To prove the second assertion it is sufficient to show that $\ker(\lambda I_Y - A|_Y) = 0$. If $w \in Y$ and $(\lambda I_Y - A|_Y)w = 0$, then $w \in K_1 \subset K_r$ so $w \in K_r \cap Y = \{0\}$. ■

2 Completeness of the generalized eigenspaces

Theorem 2.1 *Suppose that A is a linear transformation from a finite dimensional complex vector space $\mathbb{V}$ to itself with characteristic polynomial*

$$\det(zI - A) \;=\; \Pi_{j=1}^{k}(z-\lambda_j)^{m_j}, \qquad \lambda_j \;\text{ distinct}\,, \qquad \sum m_j = \dim \mathbb{V}\,.$$

Denote by X_j the generalized eigenspace associated to λ_j.

i. $\dim X_j \;=\; m_j$. *Therefore the index r_j of the generalized eigenspace of λ_j is $\leq m_j$ and*

$$X_j \;:=\; \ker\,(A-\lambda_j I)^{r_j} \;=\; \ker\,(A-\lambda_j I)^{m_j}\,.$$

ii. $\mathbb{V} \;=\; X_1 \,\oplus\, X_2 \,\oplus \cdots \oplus X_k\,.$

Proof. **i.** For ease of reading fix one of the λ_j and call it λ. Proposition 1.2 shows that

$$A \;=\; A|_X \,\oplus\, A|_Y\,.$$

Taking a basis for $\mathbb{V}$ whose first elements are a basis for X and last elements a basis for Y. Abusing notation with $A|_X$ and $A|_Y$ also denoting the matrices of the restrictions yields block matrices

$$A \;=\; \begin{pmatrix} A|_X & 0 \\ 0 & A|_Y \end{pmatrix}, \qquad zI \;-A \;=\; \begin{pmatrix} zI_X - A|_X & 0 \\ 0 & zI_Y - A|_Y \end{pmatrix}.$$

Therefore for all z,

$$\det(zI - A) \;=\; \det(zI_X - A|_X)\;\det(zI_Y - A|_Y)\,. \tag{2.1}$$

Propostion 1.3 shows that λ is the only eigenvalue of $A|_X$, so

$$\det(zI_X - A|_X) \;=\; (z-\lambda)^{\dim X}\,.$$

The same proposition shows that λ is not an eigenvalue of $A|_Y$, so

$$\det(\lambda I_Y - A|_Y) \;\neq\; 0\,.$$

Therefore (2.1) shows that the multiplicity of the root λ is equal to $\dim X$. Since the mulitplicity of the root λ is equal to m, one has

$$m \;=\; \dim X\,.$$

Estimate (1.1) with $\ell = r$ yields $\dim X = \dim K_r \geq r$ proving the estimate for the index.

ii. Using **i** yields

$$\dim X_1 \times X_2 \times \cdots \times X_k \;=\; \sum \dim X_j \;=\; \sum_1^k m_j \;=\; \dim \mathbb{V}\,.$$

The map

$$X_1 \times X_2 \times \cdots \times X_k \ni (x_1, x_2, \dots, x_k) \quad\mapsto\quad x_1 + x_2 + \cdots + x_k \;\in\; \mathbb{V}$$

is a linear map of spaces of the same dimension. To prove the direct sum assertion of the Theorem it is sufficient to show that it is injective.

If $x_1 + \cdots + x_k = 0$, multiplying by $\Pi_{j\neq\mu}(A-\lambda_j I)^{m_j}$ annihilates all the summands except the x_μ term to yield,

$$\Pi_{j\neq\mu}(A-\lambda_j I)^{m_j} x_\mu \;=\; 0\,.$$

Proposition 1.3 shows λ_μ is the only eigenvalue of $A|_{X_\mu}$, so for $j \neq \mu$, $A - \lambda_j I$ is an invertible map of X_μ to itself. Therefore $\Pi_{j\neq\mu}(A-\lambda_j I)^{m_j}$ is invertible from X_μ to itself. It follows that $x_\mu = 0$. Since this is true for each μ one has $x_1 = x_2 = \cdots = x_k = 0$ proving injectivity. ∎

Exercise 2.1 *Suppose that A is a real matrix as in Exercise 1.1 and that λ and $\overline{\lambda}$ is a pair of complex conjugate eigenvalues with generalized eigenspaces X and $\overline{X}$. If $m \geq 1$ and $w_1, w_2, \dots, w_m$ is a basis for X, prove that*

$$\operatorname{Re} w_1\,,\ \operatorname{Im} w_1\,,\ \operatorname{Re} w_2\,,\ \operatorname{Im} w_2\,,\ \dots\,,\ \operatorname{Re} w_m\,,\ \operatorname{Im} w_m$$

is a basis for $X \oplus \overline{X}$. **Hint.** Show that they are independent and that their number matches the dimension.

Exercise 2.2 *The polynomial $p(z) := \det(zI - A)$ is called the characteristic polynomial of A. Prove the Cayley-Hamilton Theorem asserting that $p(A) = 0$.* **Hint.** *Show that if v is a member of one of the generalized eigenspaces X_j, then $p(A)v = 0$.*

Exercise 2.3 *A different proof of the Cayley-Hamilton Theorem uses the fact that the set of matrices whose characteristic polynomial has no repeated roots is dense in the set of all matrices.* **i.** *Show that the Cayley-Hamilton Theorem is true for this dense set of matrices by evaluating $p(A)v$ on a basis of eigenvectors v.* **ii.** *Then approximate a general matrix A by matrices A_n whose polynomials have no repeated roots. Pass to the limit $n \to \infty$ in the Cayley-Hamilton Theorem applied to A_n.* **Hint.** Show that the characteristic polynomial p_n of A_n converges to the characteristic polynomial of A.

3 General solution to constant coefficient systems

Theorem 3.1 *Suppose that A, λ_j, m_j, and X_j are as above. If $v \in X_j$ then the solution of the initial value problem*

$$X' = AX, \qquad X(0) = v$$

is given by

$$X(t) = e^{\lambda_j t}\left[I + t(A-\lambda_j I) + \frac{t^2(A-\lambda_j I)^2}{2!} + \cdots + \frac{t^{m_j-1}(A-\lambda_j I)^{m_j-1}}{(m_j-1)!}\right] v .$$

Proof. Write $A = \lambda_j I + (A - \lambda_j I)$ as the sum of commuting transformations and compute

$$e^{At} v = e^{\lambda_j t I} e^{At - \lambda_j t I} v = e^{\lambda_j t I}\left[\sum_{k=0}^{\infty} \frac{t^k (A-\lambda_j I)^k}{k!} v\right].$$

The result follows because $e^{\lambda_j t I} = e^{\lambda_j t} I$ and the infinite sum terminates at power $k = m_j - 1$ since $v \in X_j$. ∎

To construct a general solution it suffices to find a basis v_j each element of which is a generalized eigenvector and to use the Theorem to compute the solution $\Phi_j(t)$ with initial value v_j. This is spelled out in more detail in the Multiple Roots Handout.

4 Diagonalisability of symmetric matrices

We derive the fact that symmetric matrices have an orthonormal basis of eigenvectors from the result of the preceding section. Recall that the analogue in a general scalar product space of symmetric matrices are the linear transformations which satisfy

$$\forall\, v, w, \qquad \langle Av, w \rangle = \langle v, Aw \rangle$$

where $\langle\ ,\ \rangle$ denotes the scalar product.

Example 4.1 *If $\mathbb{V} = \mathbb{C}^n$ the standard scalar product is*

$$\langle v, w \rangle = \sum x_j \, \overline{y}_j ,$$

and the corresponding symmetric linear transformations are the matrices satisfying

$$A_{ij} = A^*_{ji} .$$

They are hermitian symmetric. For real matrices that reduces to symmetry.

Theorem 4.2 *Suppose that $\mathbb{V}$ is a complex scalar product space and A is a symmetric linear transformation from $\mathbb{V}$ to itself.*

i. *The eigenvalues of A are real.*

ii. *For each eigenvalue, the index $r = 1$ that is $K_\ell = K_1$ for all ℓ so every generalized eigenvector is an eigenvector.*

iii. *Eigenvectors with distinct eigenvalues are orthogonal.*

Proof. **i.** If λ is an eigenvalue, choose a unit eigenvector v to find,

$$\lambda = \lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Av, v\rangle = \langle v, Av\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\langle v, v\rangle = \overline{\lambda}\,.$$

ii. Must show that $K_2 = K_1$. That is if $(A-\lambda)^2 v = 0$ then $(A - \lambda I)v = 0$. Compute

$$\|(A-\lambda I)v\|^2 = \big\langle (A-\lambda I)v, (A-\lambda I)v\big\rangle \;=\; \big\langle (A-\lambda I)^2 v, v\big\rangle = \langle 0, v\rangle = 0\,.$$

iii. If $Av = \lambda_1 v$ and $Aw = \lambda_2 w$ with $\lambda_1 \neq \lambda_2$ must show that $\langle v, w\rangle = 0$. Compute using the fact that the eigenvalues are real,

$$\lambda_1\langle v, w\rangle = \langle \lambda_1 v, w\rangle = \langle Av, w\rangle = \langle v, Aw\rangle = \langle v, \lambda_2 w\rangle = \lambda_2\langle v, w\rangle\,.$$

Therefore $(\lambda_1 - \lambda_2)\langle v, w\rangle = 0$ showing that $\langle v, w\rangle = 0$. ■

5 Application to stability for $X' = AX$

The Spectral Decomposition Theorem yields characterizations of asymptotic stability and stability of the equilibrium solution $X = 0$ of $X' = AX$. In both cases the description in terms of the spectrum of A is the criterion most often employed.

5.1 Asymptotic stability

The next results require that one choose a norm $\|\cdot\|$ on $\mathbb{V}$. When $\mathbb{V} = \mathbb{C}^N$ it is sometimes natural to choose the Euclidean norm. The four equivalent conditions in the theorem are ordered from strongest to weakest.

Theorem 5.1 *Suppose that $\mathbb{V}$ is a finite dimensional complex normed vector space and that $A : \mathbb{V} \to \mathbb{V}$ is linear. For the differential equation $X' = AX$ the following are equivalent.*

1. *There are positive constants K and ρ so that for all $t \geq 0$*

$$\|e^{At}\| \leq K\, e^{-\rho t}\,. \tag{5.1}$$

2. 0 *is an asymptotically stable equilibrium.*

3. *Every solution $X(t)$ converges to 0 as $t \to \infty$.*

4. *All the eigenvalues of A have strictly negative real part.*

Proof. It suffices to show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$.

Exercise 5.1 *Prove the easiest steps* $1 \Rightarrow 2 \Rightarrow 3$.

$3 \Rightarrow 4$. The implication $3 \Rightarrow 4$ is equivalent to contrapositive $\sim 4 \Rightarrow \sim 3$ where $\sim$ denotes the denial of.

The statement ~ 4 means that there is an eigenvalue λ with real part ≥ 0. Choose an eignevector $v \neq 0$. Then $X(t) := e^{\lambda t} v$ is a solution that does not tend to zero as $t \to \infty$ proving ~ 3.

$4 \Rightarrow 1$. This is the important implication. Denote by X_j the generalized eigenspaces of A. Any v has a unique expansion $v = \sum v_j$ with $v_j \in X_j$. By the triangle inequality, it follows that

$$\|v\| \;\leq\; \sum \|v_j\|\,.$$

On the other hand the map $v \mapsto v_j$ is a linear map from $\mathbb{V}$ to $X_j \subset \mathbb{V}$ so there is a constant C_j so that $\|v_j\| \;\leq\; C_j\,\|v\|$.

Exercise 5.2 *Expanding v as a sum of elements in the generalized eigenspaces show that it suffices to show that for each generalized eigenspace X_j associated to A there are positive constants ρ_j and K_j so that for $v \in X_j$*

$$\|e^{tA}v\| \;\leq\; K_j\, e^{-\rho_j t}\,\|v\|\,. \tag{5.2}$$

Denote by $m_j \geq 1$ the multiplicity of λ_j as a root of the characteristic polynomial of A. Then for $v \in X_j$

$$e^{At}v \;=\; e^{\lambda_j t}\Big[I + t(A-\lambda_j I) + \frac{t^2(A-\lambda_j I)^2}{2!} \;+\;\cdots\;+\; \frac{t^{m_j-1}(A-\lambda_j I)^{m_j-1}}{(m_j-1)!}\Big]v\,.$$

Since λ_j has strictly negative real part we can choose $0 < \rho_j < |\mathrm{Re}\,\lambda_j|$ then for each μ there are constants $C(\mu,\rho_j)$ so that for $t \geq 0$

$$|t^\mu\, e^{\lambda_j t}| \;\leq\; C\, e^{-\rho_j t}\,.$$

The desired estimate (5.2) then follows from the triangle inequality. ∎

5.2 Stability

Theorem 5.2 *For the linear equation $X' = AX$, the following are equivalent.*

1. *There is a constant K so that for all $t \geq 0$*

$$\|e^{tA}\| \;\leq\; K\,.$$

2. *0 is a stable equilibrium.*

3. *Every solution $X(t)$ is uniformly bounded for $t \geq 0$.*

4. *Each eigenvalue of A has real part ≤ 0. In addition if λ is a purely imaginary eigenvalue and m its multiplicity, then $\ker(A-\lambda I)$ has dimension m.*

Proof. The proof is by showing that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$.

Exercise 5.3 *Prove $1 \Rightarrow 2 \Rightarrow 3$.*

$3 \Rightarrow 4$ is equivalent to $\sim 4 \Rightarrow \sim 3$. If 4 is violated either there is an eigenvalue with strictly negative real part or an eigenvalue in the imaginary axis and a vector $v \neq 0$ so that

$$(A-\lambda I)v \neq 0\,, \qquad \text{and} \qquad (A-\lambda I)^2 v = 0\,.$$

In the first case if v is an eigenvector then $e^{\lambda t}v$ is an unbounded solution. In the second case

$$e^{\lambda t}\Big[I + t(A-\lambda I)\Big]\,v$$

is an unbounded solution. In both cases we have ~ 3.

$4 \Rightarrow 1$. It is sufficient to show that for each generalized eigenspace X_j there is a constant K_j so that for all $v \in X_j$ and $t \geq 0$,

$$\|e^{tA}v\| \;\leq\; K_j\,\|v\|\,. \tag{5.3}$$

If the eigenvalue for X_j has strictly negative real part, then Theorem 5.1 applied in the vector space X_j implies that there are positive K_j and ρ_j so that $\|e^{tA}\| \leq K_j e^{-\rho_j t}$. This implies (5.3)

If the eigenvalue is purely imaginary, then X_j has dimension m equal to the dimension of the kernel so for $v \in X_j$, $Av = \lambda_j v$. Therefore for those v, $e^{tA}v = e^{t\lambda_j}v$. Since λ_j is purely imaginary $e^{\lambda_j t}$ has modulus equal to one so

$$\|e^{tA}v\| \;=\; \|e^{t\lambda_j}v\| \;=\; |e^{t\lambda_j}|\,\|v\| \;=\; \|v\|\,.$$

This is the desired estimate (5.3) with $K_j = 1$. ■

6 Quadratic forms decreasing on orbits

6.1 General result

This section gives a generalization of two easy examples. The decreasing quadratic form is used (not in these notes) to prove the asymptotic stability of equilibria of nonlinear problems whose linearization is asymptotically stable.

Example 6.1 **i.** *For a real 2×2 system whose phase plane is a spiral sink the orbits are of the form e^{at} times elliptical orbits where $a < 0$ is the real part of the complex conjugate eigenvalues. The positive quadratic form whose level sets define the ellipses is then decreasing on non zero orbits.*

ii. *For a 2×2 system with two distinct negative real eigenvalues λ_j and eigenvectors V_j define new coordinates α_j by*

$$X \;=\; \alpha_1 V_1 \;+\; \alpha_2 V_2\,.$$

A function $X(t)$ satisfies the differential equation if and only if $\alpha_j(t) = c_j e^{\lambda_j t}$. Therefore the quadratic form

$$\sum_j |\alpha_j|^2 \tag{6.1}$$

decreases on orbits.

In both cases, the vector field is transverse to and points into the ellipsoids that are level sets of the positive definite quadratic form.

Definition 6.1 *A scalar product on a complex vector space $\mathbb{V}$ is a mapping $Q : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ satisfying for all X, Y, and Z in $\mathbb{V}$ and $a \in \mathbb{C}$,*

i. $Q(X+Y,Z) \;=\; Q(X,Z) + Q(Y,Z)$,

ii. $Q(aX,Z) = aQ(X,Z)$,

iii. $Q(X,Z) \;=\; \overline{Q(Z,X)}$,

iv. $Q(X,X) > 0$ *if* $X \neq 0$.

Exercise 6.1 *If Q is a scalar product show that $Q(0,Y) = 0$, $Q(X,aY) = \overline{a}\,Q(X,Y)$, and $Q(X,Y+Z) = Q(X,Y) + Q(X,Z)$.*

Example 6.2 *If v_j, $1 \le j \le \dim \mathbb{V}$ is a basis then $u, v \in \mathbb{V}$ have unique expansions*

$$u \;=\; \sum \alpha_j\, v_j\,, \qquad v \;=\; \sum \beta_j\, v_j\,.$$

Then $Q(u,v) := \sum \alpha_j \overline{\beta}_j$ is a scalar product. With respect to this scalar product the basis is orthonormal.

Example 6.3 *Suppose that $\mathbb{V}$ is a complex finite dimensional vector space, $A : \mathbb{V} \to \mathbb{V}$ is linear and has a basis of eigenvectors $\{v_j\}$, and, all eigenvalues have strictly negative real part. Then the scalar product in the preceding example decreases on orbits of $X' = AX$.*

This example includes the preceding two as special cases and almost yields the general result. It misses only the case of A that have non trivial generalized eigenspaces.

Theorem 6.4 *(Lyapunov) Suppose that $\mathbb{V}$ is a finite dimensional complex vector space and that $A : \mathbb{V} \to \mathbb{V}$ is a linear transformation whose eigenvalues have strictly negative real part. Then there is a scalar product $Q(X,Y)$ on $\mathbb{V}$ and a constant $c > 0$ to that for all solutions of $X' = AX$ one has*

$$\frac{dQ(X(t),X(t))}{dt} \;\le\; -c\,Q(X(t),X(t))\,. \tag{6.2}$$

In particular $Q(X(t),X(t))$ is strictly decreasing on non zero orbits.

Geometric interpretation. Orbits cross from ellipsoids $Q = const$ to ellipsoids with smaller constants.

The proof uses the following characterization of transformations whose only eigenvalue is 0.

Lemma 6.1 *If $\mathbb{W}$ is a finite dimensional complex vector space and $B : \mathbb{W} \to \mathbb{W}$ is linear then the following are equivalent.*

i. *Zero is the only eigenvalue of B.*

ii. $B^{\dim \mathbb{W}} = 0$.

iii. *For every $\varepsilon > 0$ there is a scalar product Q on $\mathbb{W}$ so that for all $w \in \mathbb{W}$,*

$$Q(Bw,Bw) \;\le\; \varepsilon^2\, Q(w,w)\,. \tag{6.3}$$

Equivalently, in the norm defined by Q, $\|Bw\|_Q \le \varepsilon \|w\|_Q$.

Proof. **i.$\Rightarrow$ii**. Since 0 is the only eigenvalue, the only generalized eigenspace is the one associated to $\lambda = 0$. The multiplicity of the eigenvalue must be equal to $\dim \mathbb{W}$. The Spectral Theorem for General Matrices implies **ii**.

ii.$\Rightarrow$iii. This is the difficult step. Begin by choosing any scalar product on $\mathbb{W}$. Denote the scalar product by $(\,\cdot\,,\,\cdot\,)$ and by m the dimension of $\mathbb{W}$. With a thunder bolt of creativity define a new scalar product Q by

$$\begin{aligned} Q(X,W) &:= \sum_{j=0}^{m-1} \big(\Gamma^j\, B^j X\,,\, \Gamma^j\, B^j W\big) \\ &:= \big(X\,,\,W\big) + \big(\Gamma\, BX\,,\,\Gamma\, BW\big) + \big(\Gamma^2\, B^2 X\,,\,\Gamma^2\, B^2 W\big) + \cdots + \big(\Gamma^{m-1}\, B^{m-1} X\,,\,\Gamma^{m-1}\, B^{m-1} W\big). \end{aligned}$$

with large positive Γ to be chosen later.

Exercise 6.2 *Verify that if* $\Gamma \geq 0$ *then* Q *is a scalar product.*

By definition

$$Q(w,w) = (w,w) + (\Gamma Bw, \Gamma Bw) + \cdots + \big(\Gamma^{m-1} B^{m-1} w\,,\, \Gamma^{m-1} B^{m-1} w\big).$$

Then,

$$Q(Bw,Bw) = (Bw,Bw) + (\Gamma Bw, \Gamma Bw) + \cdots + \big(\Gamma^{m-2} B^{m-1} w\,,\, \Gamma^{m-2} B^{m-1} w\big) + \big(\Gamma^{m-1} B^{m} w\,,\, \Gamma^{m-1} B^{m} w\big).$$

Mulitply by Γ^2 and use $B^m = 0$ to find

$$\begin{aligned} \Gamma^2 Q(Bw,Bw) &= (\Gamma Bw, \Gamma Bw) + \cdots + \big(\Gamma^{m-1} B^{m-1} w\,,\, \Gamma^{m-1} B^{m-1} w\big) \\ &= Q(w,w) \; - \; (w,w) \; \leq \; Q(w,w)\,. \end{aligned}$$

Choosing Γ so large that $1/\Gamma \leq \varepsilon$ proves **iii**.

iii.⇒i. If λ is an eigenvalue, choose an eigenvector w. Using **iii** compute

$$|\lambda| \|w\| = \|\lambda\, w\| = \|Bw\| \leq \varepsilon \|w\|\,.$$

Therefore $|\lambda| \leq \varepsilon$. Since this is true for all $\varepsilon > 0$ it follows that $\lambda = 0$. This completes the proof of the Lemma. ■

Proof of Theorem. Step I. Reduction to the case of $(A - \lambda I)^m = 0$**.** Decompose $\mathbb{V} = \oplus X_j$ as a direct sum of generalized eigenspaces. The X_j are invariant by A and also by the flow of the differential equation.

To construct Q it is sufficient to construct scalar products Q_j on X_j that decrease under the flow of the differential equation restricted to X_j. Then $Q(X,X) := \sum_j Q_j(X_j, X_j)$ serves for the original problem.

Thus it is sufficient to consider $A|_{X_j}$. That restriction has one eigenvalue and satisfies $(A|_{X_j} - \lambda_j I_{X_j})^{m_j} = 0$. This reduces to the case of transformations A so that there is a $\lambda \in \{\operatorname{Re} z < 0\}$ and an $m \geq 1$ so that $(A - \lambda I)^m = 0$. The remainder of the proof treats that case.

Step II. Use Lemma 6.1. Define $B = A - \lambda I$ with $B^m = 0$. If $X' = AX$, then $Y = e^{-\lambda t} X$ satisfies

$$Y' = \Big(e^{-\lambda t} X\Big)' = e^{-\lambda t} X' - \lambda e^{-\lambda t} X = e^{-\lambda t}\Big(AX - \lambda X\Big) = BY\,.$$

The strategy is to find a scalar product that grows slowly under $Y' = BY$ so that $X = e^{\lambda t} Y$ decays because of the exponential factor.

For $\varepsilon > 0$ to be chosen later, choose Q as in the Lemma. A product rule yields

$$\frac{d}{dt} Q(Y(t), Y(t)) = \; Q(Y', Y) + Q(Y, Y') \; = \; Q(BY, Y) + Q(Y, BY)\,. \tag{6.4}$$

The Cauchy-Schwartz inequality for the scalar product Q and then (6.3) yield

$$\begin{aligned} |Q(BY,Y)| \; &= \; |Q(Y,BY)| \; \leq \; Q(BY,BY)^{1/2} Q(Y,Y)^{1/2} \\ &\leq \; \big(\varepsilon^2 Q(Y,Y)\big)^{1/2} Q(Y,Y)^{1/2} \; = \; \varepsilon\, Q(Y,Y)\,. \end{aligned}$$

Therefore

$$\frac{d}{dt} Q(Y(t), Y(t)) \; \leq \; 2\,\varepsilon\, Q(Y(t), Y(t))\,. \tag{6.5}$$

Step III. Endgame. Since $|e^{\lambda t}| = e^{\mathrm{Re}\,\lambda t}$ one has

$$Q(X,X) \;=\; Q(e^{\lambda t}Y\,,\,e^{\lambda t}Y) \;=\; e^{\lambda t}\,\overline{e^{\lambda t}}\,Q(Y,Y) \;=\; |e^{\lambda t}|^2\,Q(Y,Y) \;=\; e^{2\mathrm{Re}\,\lambda t}\,Q(Y,Y)\,. \qquad (6.6)$$

Differentiate to find

$$\frac{d}{dt}\Big(e^{2\mathrm{Re}\,\lambda t}\,Q(Y(t),Y(t))\Big) \;=\; 2\mathrm{Re}\,\lambda\; e^{2\mathrm{Re}\,\lambda t}\;Q(Y,Y) + e^{2\mathrm{Re}\,\lambda t}\;\frac{d}{dt}Q(Y(t),Y(t))\,.$$

Using (6.6) yields

$$\frac{d}{dt}\,Q(X,X) \;\le\; (2\,\mathrm{Re}\,\lambda)\;Q(X,X) \;+\; 2\,\varepsilon\,e^{2\mathrm{Re}\,\lambda t}\;Q(Y,Y) \;=\; \Big(2\,\mathrm{Re}\,\lambda + 2\,\varepsilon\Big)Q(X,X)\,. \qquad (6.7)$$

Choose ε so small that $\varepsilon \;<\; |\mathrm{Re}\,\lambda|$. Then the coefficient in front of $Q(X,X)$ is strictly negative. This complete the proof of the Theorem. ■

Example 6.5 *Consider the special case*

$$A = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}, \qquad \lambda = -1, \qquad B = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}, \qquad m = 2\,.$$

Denote by $(\;,\;)$ *and* $\|\;\|$ *the euclidean scalar product and norm. The preceding analysis shows that for* Γ *sufficiently large,*

$$\|X\|^2 \;+\; \Gamma^2 \left\| \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} X \right\|^2 \qquad (6.8)$$

decreases on orbits.

Exercise 6.3 *Continue this example by finding the constant* Γ_0 *so that the quadratic form in* (6.8) *is strictly decreasing on non zero orbits for* $\Gamma > \Gamma_0$ *and not for* $\Gamma \le \Gamma_0$. **Hint.** Write the time derivative as a quadratic form in the coordinates (x_1, x_2). Check negativity using the fact that $ax_1^2 + bx_1x_2 + cx_2^2$ is strictly negative definite if and only if $a < 0$, and $b^2 - 4ac < 0$. The proof of this necessary and sufficient condition for definiteness goes as follows. The necessity of $a < 0$ comes from considering $x_2 = 0$. If $b^2 - 4ac \ge 0$ then one can find a point $(x_1, x_2) \ne 0$ where the form vanishes so $b^2 - 4ac < 0$ is also necessary. For sufficiency complete squares to find

$$ax_1^2 + bx_1x_2 + cx_2^2 \;=\; a\Big(x_1 + \frac{b}{2a}x_2\Big)^2 \;+\; \Big(c - \frac{b^2}{4a}\Big)x_2^2 \;=\; a\Big(x_1 + \frac{b}{2a}x_2\Big)^2 + \frac{1}{4a}(4ac - b^2)x_2^2\,.$$

Exercise 6.4 *Show that if there is a scalar product* $Q(X,X)$ *that is strictly decreasing on nonzero orbits of* $X' = AX$, *then the eigenvalues of* A *have strictly negative real part.* **Hint.** Consider the solution $X(t) = e^{\lambda t}\mathbf{v}$. You must be careful about complex numbers and complex scalar products. **Discussion.** The assertion is the converse of the Theorem.

Exercise 6.5 *For the damped linear spring,* $x'' + x' + x = 0$ *show that there is no choice of the constant* $A > 0$ *so that the quadratic form* $Q := (x')^2 + Ax^2$ *is strictly decreasing on orbits.*

Exercise 6.6 *Continuing the damped linear spring show that for* $\varepsilon > 0$ *sufficiently small the quadratic form* $R := (x')^2 + x^2 + \varepsilon\, x'\, x$ *is strictly positive definite and strictly decreasing on orbits.*

Exercise 6.7 An *incorrect proof* of the Theorem is the following. Given any $\varepsilon > 0$ one can Choose $\widetilde{A}$ with $\|A - \widetilde{A}\| < \varepsilon$ so that $\widetilde{A}$ has distinct eigenvalues with strictly negative real part. Easily construct a scalar product $\widetilde{Q}$ that satisfies (6.2) on orbits of $X' = \widetilde{A}\, X$ as in Example 6.3. It follows that there is a $\delta > 0$ so that if $\|B - \widetilde{A}\| < \delta$, then $\widetilde{Q}$ decreases on orbits of $X' = BX$. Since A is as close to $\widetilde{A}$ as one likes, $\widetilde{Q}$ is the desired scalar product. *Explain the error in the last sentence.*

6.2 Reality

In this section we show that when A is a real matrix the scalar product Q can be chosen real. Even in the general case, though Q may be complex, the decreasing functional $Q(X(t), X(t))$ is real valued and that is enough.

Definition 6.2 *A linear subspace $\mathbb{V} \subset \mathbb{C}^N$ is called real when each of the following equivalent conditions hold.*

i. *The complex conjugate of each vector $v \in \mathbb{V}$ belongs to $\mathbb{V}$.*

ii. *The real and imaginary parts of each vector $v \in \mathbb{V}$ belong to $\mathbb{V}$.*

iii. *$\mathbb{V}$ has a basis of real eigenvectors.*

Exercise 6.8 *Prove the easy implications* **i** $\Leftrightarrow$ **ii** *and* **iii** $\Rightarrow$ **i**.

To see that **iii** is implied by the others, reason as follows. Choose v_α a finite spanning set of complex vectors of $\mathbb{V}$. Denote by w_β the set of real and imaginary parts of these spanning vectors. The w_β is a spanning set of real vectors with twice as many elements as a basis.

Consider the finite family of subsets of the w_β. Some, for example a subset consisting of one nonzero element, are linearly independent. Denote by k the size of the largest of these independent subsets. Choose $u_1, \dots, u_k$ one of these largest independent sets. Then each w_β is in the span of the u_j. Otherwise, adjoining w_β would yields a larger independent set. Therefore the u_k is a real basis.

Exercise 6.9 *Give details of the last two assertions.*

Exercise 6.10 *Suppose that A is a real $N \times N$ matrix, $\lambda \in \mathbb{R}$ is a real eigenvalue, and X_λ is its generalized eigenspace. Show that X_λ is a real subspace of $\mathbb{C}^N$.*

Example 6.6 **i.** *Suppose that $\mathbb{U} \subset \mathbb{C}^N$ is a linear subspace. Show that $\overline{\mathbb{U}}$ is a linear subspace.*

ii. *Show that* $\text{Span}\,\{\mathbb{U} \cup \overline{\mathbb{U}}\}$ *is a* **real** *linear subspace.*

iii. *If in addition $\mathbb{U} \cap \overline{\mathbb{U}} = \{0\}$ show that the span has dimension equal to twice the dimension of $\mathbb{U}$.*

Example 6.7 *Suppose that A is a real $N \times N$ matrix, $\lambda \in \mathbb{C}$ is an eigenvalue that is* **not** *real. Denote by X_λ the generalized eigenspace associated to λ and $\overline{X_\lambda}$ the complex conjugate. Exercise 1.1 shows that $\overline{X_\lambda}$ is the generalized eigenspace associated to $\overline{\lambda}$. Exercise 2.1 shows that $X_\lambda \oplus X_{\overline{\lambda}}$ is a real vector space.*

Exercise 6.11 *If Q is a scalar product defined on a real subspace $\mathbb{U} \subset \mathbb{C}^N$ show that the following are equivalent.*

i. *For all $v, w \in \mathbb{U}$, $Q(\overline{v}, \overline{w}) = \overline{Q(v,w)}$.*

ii. *For all real $v, w \in \mathbb{U}$, $Q(v,w) \in \mathbb{R}$.*

iii. *For all real $v \in \mathbb{U}$, $Q(v,v) \in \mathbb{R}$.*

Definition 6.3 *When these conditions are satisfied, the scalar product is called* **real**.

Example 6.8 *The standard scalar product on $\mathbb{C}^N$ given by $\sum_j x_j \overline{y_j}$ is real.*

Example 6.9 *If $\mathbb{C}^N$ is the direct sum of* **real** *subspaces U_α, and for each α, Q_α is a* **real** *scalar product on U_α, then $\sum_\alpha Q_\alpha$ is a real scalar product on $\mathbb{C}^N$.*

Corollary 6.10 *If $\mathbb{V} = \mathbb{C}^N$ and A is a* **real** *matrix with eigenvalues of strictly negative real part, then there is a* **real** *scalar product Q so that $Q(X,X)$ is strictly decreasing on orbits of $X' = AX$.*

The construction in the preceding subsection does not usually yield a real Q. Typically the generalized eigenspace associated to a complex eigenvalue has no nonzero real vectors in it. To achieve reality one must link the constructions on the generalized eigenspaces associated to λ and $\overline{\lambda}$.

Proof. Step I. Real direct sum decomposition. Denote by $\lambda_1, \dots, \lambda_J$ the distinct real eigenvalues of A and X_{λ_j} their generalized eigenspaces. Exercise 6.10 shows that each of the X_{λ_j} is a real subspace.

Denote by $\mu_1, \dots, \mu_K$ the distinct eigenvalues of A *with strictly positive imaginary part.* Denote by X_{μ_k} the generalized eigenspace. Exercise 2.1 shows that the generalized eigenspace associated to $\overline{\mu_k}$ is equal to $\overline{X_{\mu_k}}$. Since the generalized eigenspaces associated to μ_j and $\overline{\mu}_k$ intersect exactly at 0 the direct sum $Y_k := X_{\mu_k} \oplus \overline{X_{\mu_k}}$ is a real subspace with dimension equal to twice the dimension of X_{μ_k}.

The general spectral theorem implies that one has the direct sum decomposition in real subspaces

$$\mathbb{C}^N = X_{\lambda_1} \oplus \cdots \oplus X_{\lambda_J} \oplus Y_1 \oplus \cdots \oplus Y_K .$$

Each of the summands is invariant under A and therefore under the flow of the differential equation $X' = AX$.

Thanks to Example 6.9 it is sufficient to construct real decreasing scalar products on each of the direct summands.

Step II. Decreasing scalar product on the X_j. The construction of Q yields a real scalar product provided that the beginning scalar product $(\cdot, \cdot)$ on X_j is real.

A real scalar product on X_j is constructed as follows. Choose a real basis v_α of X_j. Then the unique scalar product so that the v_α form an orthonormal basis is such a real scalar product. If the coordinates of vectors v, w in this basis are x_α and y_α respectively, then the scalar product is given by $\sum_\alpha x_\alpha \overline{y_\alpha}$.

Step III. Decreasing scalar product on the Y_k. The proof constructs a decreasing scalar product on X_{μ_k}. Call that scalar product P_k. Define a unique scalar product Q_k on $Y_k := X_{\mu_k} \oplus \overline{X_{\mu_k}}$ so that

• the two direct summands are orthogonal with respect to Q,
• Q restricted to X_{μ_k} is equal to P, and,
• for v, w belonging to $\overline{X_{\mu_k}}$,

$$Q_k(v,w) \; := \; \overline{P(\overline{v},\overline{w})} \, .$$

Then Q_k is a real scalar product on Y_k strictly decreasing on orbits.

Exercise 6.12 *Verify the last sentence.* ■

7 Floquet theory

Floquet theory is the study of linear equations with coefficients that are periodic in t,

$$X' \; = \; A(t)\, X \, , \qquad A(t+T) = A(t) \, . \tag{7.1}$$

Suppose that $A(t)$ is continuous. Define the map M by

$$M\, X_0 \; := \; X(T)$$

where $X(t)$ is the unique solution of $X' = A(t)X$ with $X(0) = X_0$. The linear transformation M is the Poincaré map or time T map of the system.

Thanks to the periodicity of the coefficients, $X(nT) = M^n X(0)$ so studying the long time behavior of orbits is equivalent to studying the powers of the linear transformation M.

Definition 7.1 *M is called the* **Floquet map** *and its eigenvalues λ_j are called* **Floquet multipliers**.

The next result describes the systems for which the zero solution is asymptotically stable.

Theorem 7.1 *If $\mathbb{V}$ is a finite dimensional complex vector space and $M : \mathbb{V} \to \mathbb{V}$ is linear, then the following are equivalent.*

i. *For any vector v, $\lim_{n\to\infty} M^n v = 0$.*

ii. *The eigenvalues of M all have modulus strictly less than 1.*

iii. *If $1 > a > \max_j |\lambda_j|$ then there is a scalar product Q so that in the norm defined by Q,*

$$\|Mv\|_Q \; \leq \; a\, \|v\|_Q \qquad \text{for all } v \in \mathbb{V} \, .$$

iv. *There are positive constants c, α so that*

$$\|M^n\| \leq c\, e^{-\alpha n} \, .$$

Proof. Prove the four implications, **i** ⇒ **ii** ⇒ **iii** ⇒ **iv** ⇒ **i**.

Exercise 7.1 *Prove* **iv** ⇒ **i** ⇒ **ii**.

Proof that ii ⇒ iii. Denote by X_j the generalized eigenspaces of λ_j. It is sufficient to show that there are scalar products Q_j on X_j so that

$$\|Mv\|_{Q_j} \;\le\; a\,\|v\|_{Q_j} \qquad \text{for all } v \in X_j\,.$$

Thus it suffices to consider the case of only one eigenvalue. We do that and drop the subscripts j.

Choose $0 < \varepsilon < a - |\lambda|$. Lemma 6.1 applied to $B := M - \lambda I$ implies that there is a scalar product Q so that for all v

$$\|(M - \lambda I)v\|_Q \;\le\; \varepsilon\,\|v\|_Q\,.$$

Then

$$\|Mv\|_Q = \|\lambda v + (M-\lambda I)v\|_Q \le \|\lambda v\|_Q + \|(M-\lambda I)v\|_Q \le |\lambda|\,\|v\|_Q + \varepsilon\|v\|_Q = (|\lambda|+\varepsilon)\,\|v\|_Q \;<\; a\,\|v\|_Q\,.$$

Proof that iii ⇒ iv. Choose $a < 1$ and Q as in **iii**. Choose $\alpha > 0$ so that $e^{-\alpha} = a$. By induction on n one has for all v and integers $n \ge 0$,

$$\|M^n v\|_Q \;\le\; a^n \|v\|_Q \;=\; e^{-\alpha n}\|v\|_Q\,.$$

There are constants $0 < c_1 < C_2 < \infty$ so that for all vectors v

$$c_1\|v\|_Q \;\le\; \|v\| \;\le\; C_1\|v\|_Q\,.$$

Therefore

$$\|M^n v\| \;\le\; C_1\,\|M^n v\|_Q \;\le\; C_1\,e^{-\alpha n}\|v\|_Q \;\le\; (C_1/c_1)\,\|v\|\,.$$

The proof is complete. ■

Remark 7.1 i. *Condition* **iii** *is at the heart of the sufficient linearization criterion for asymptotic stability of fixed points of mappings.*

ii. *Applied to the Poincaré map it yields the linearization criterion for orbital asymptotic stability of periodic orbits of autonomous systems.*

Corollary 7.2 *Suppose that $X' = A(t)X$ is a linear system with continuous T-periodic coefficient and Floquet map M. The the following are equivalent.*

i. *All solutions $X(t)$ satisfy $\lim_{t\to\infty} X(t) = 0$.*

ii. *There are positive constants γ, α so that for all solutions and all $t \ge 0$*

$$\|X(t)\| \;\le\; \gamma\, e^{-\beta t}\|X(0)\|\,.$$

iii. *The eigenvalues of M all have modulus strictly less than one.*

Proof. **iii** ⇒ **ii** ⇒ **ii** ⇒ **iii**. Only **iii** ⇒ **ii** is difficult.

Exercise 7.2 *Prove the other two implications.*

iii ⇒ ii Denote by λ_j the eigenvalues and $\mathbb{V}_j$ the associated generalized eigenspace. The triangle inequality implies that It is sufficient to show that for each j there are positive constants γ_j, β_j so that for all $v \in \mathbb{V}_j$, the solution with $X(0) = v$ satisfies

$$\|X(t)\| \;\le\; \gamma_j\, e^{-\beta_j t} \|X(0)\|\,.$$

Denote by $\Psi(t)$ the fundamental matrix with $\Psi(0) = I$. For $t \ge 0$ write $t = nT + s$ with $0 \le s < T$ and $n \ge 0$ integer. Then

$$\Psi(t) \;=\; \Psi(s)\Psi(nT) \;=\; \Psi(s)\, M^n\,.$$

Define

$$C \;:=\; \max_{0\le s\le T} \|\Psi(s)\| \;<\; \infty\,.$$

On $\mathbb{V}_j$ M has a unique eigenvalue λ_j and $(M-\lambda_j)^{\dim \mathbb{V}_j} = 0$. Choose $0 < \varepsilon < 1 - |\lambda_j|$. Lemma 6.1 implies that there is a scalar product Q and associated norm $\|\cdot\|_Q$ so that for all $v \in \mathbb{V}_j$

$$\|Mv\|_Q \;\le\; \varepsilon\, \|v\|_Q\,.$$

Then for $v \in \mathbb{V}_j$,

$$\begin{aligned} \|Mv\|_Q \;&=\; \|\lambda_j v + (M-\lambda_j I)v\|_Q \;\le\; \|\lambda_j\, v\|_Q \;+\; \|(M-\lambda_j I)v\|_Q \\ &\le\; |\lambda_j|\|v\|_Q \;+\; \varepsilon\,\|v\|_Q \;=\; (|\lambda_j| + \varepsilon)\|v\|_Q\,. \end{aligned} \tag{7.2}$$

By construction, $|\lambda_j| + \varepsilon < 1$ so there is an $\alpha_j > 0$ so that for all $v \in \mathbb{V}_j$ and $n \ge 0$,

$$\|Mv\|_Q \;\le\; e^{-\alpha_j}\|v\|_Q \qquad \text{so} \qquad \|M^n v\|_Q \;\le\; e^{-\alpha_j n}\|v\|_Q\,.$$

Therefore

$$\|\Psi(t)v\|_Q \;\le\; C\, e^{-\alpha_j n}\|v\|_Q\,.$$

There are positive constants $c_1 < C_1$ so that for all $v \in \mathbb{V}_j$

$$c_1\,\|v\|_C \;\le\; \|v\| \;\le\; C_1\|v\|_Q\,.$$

Therefore

$$\|\Psi(t)v\| \;\le\; C_1\,\|\Psi(t)v\|_Q \;\le\; C_1\, C\, e^{-\alpha_j n}\|v\|_Q \;\le\; (C_1/c_1)\, C\, e^{-\alpha_j n}\|v\|\,.$$

For $t \ge T$ one has $n \ge t/2$ and this estimate proves the desired estimate for $v \in \mathbb{V}_j$ and $t \ge T$. The estimate for $t \le T$ follows from the continuity of $\Psi(t)$. This completes the proof. ∎

Remark 7.2 i. *Condition* **iv** *is invariant under small T-periodic perturbations of $A(t)$.*

ii. *More generally one can add a nonlinear term that is $\le C\|X\|^2$ for $\|X\|^2 \le \rho$ and the origin will silll have a basin of attraction containing a neighbhorhood of zero.*

iii. *If the flow conserves volume or a postiive definite quadratic form, then the conditions of the corollary cannot be satisfied.*

iv. *There is a corresponding result for stability in contrast to asymptotic stability that requires that the eigenvalues of M lie in the unit disk and those of modulus one must have eigenspaces spanned by eigenvenctors. That result is not stable under perturbations either linear or nonlinear.*

Remark 7.3 i. *The linearization of an equation $X' = F(X)$ at a periodic solution $\underline{X}(t)$ is an equation of the form* (7.1) *with*

$$A(t) \;:=\; D_X F(\underline{X}(t)\,.$$

The conditions of Theorem 7.1 are **never** *satisfied in this context. The reason is that $\underline{X}(t+\sigma)$ is a solution for all σ. Differentiating the equation*

$$\frac{d}{dt}\underline{X}(t+\sigma) \;=\; F(\underline{X}(t+\sigma))$$

with respect to σ then setting $\sigma = 0$ yields

$$Y'(t) \;=\; A(t)\,Y(t)\,, \qquad Y(t) \;=\; \underline{X}'(t)\,.$$

Denote $\mathbf{v} := \underline{X}'(0)$. Since $Y(t) \;=\; \Psi(t)Y(0)$ and is T-periodic one has

$$\Psi(T)\,\mathbf{v} \;=\; \Psi(T)\,Y(0) \;=\; Y(T) \;=\; Y(0) \;=\; \mathbf{v}\,.$$

Therefore $\Psi(T)$ always has 1 as an eigenvalue so can never have all eigenvalues with modulus strictly less than one.

ii. *The orbital asymptotic stability of periodic solutions of autonomous systems is analysed using the Poincaré first return map in the handout on Asymptotic Stability by Linearization. Theorem 7.1 is the key element.*

8 A spectral mapping theorem

This section computes the eigenvalues of e^A. That shows how Floquet Theory and the theory of systems with constant coefficients yield the same asymptotic stability criteria for $X' = A\,X$.

Theorem 8.1 *If $\mathbb{V}$ is a complex vector space and $A : \mathbb{V} \to \mathbb{V}$ is a linear transformation with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with multiplicities $m_1, m_2, \ldots, m_k$, then the eigenvalues of e^A are the numbers e^{λ_j}. The multiplicity of z as an eigenvalue of e^A is the sum of the multiplicities of the λ_j such that $e^{\lambda_j} = z$.*

Proof. Step I. Prove the result for an A that has only one eigenvalue λ. When A has only one eigenvalue λ, there is a μ so that $(A - \lambda I)^\mu = 0$. Let $B := A - \lambda I$ so $B^\mu = 0$.
Then

$$e^B - I \;=\; B \;+\; \frac{B^2}{2!} \;+\; \cdots \;+\; \frac{B^{\mu-1}}{(\mu-1)!} \;=\; B\,q(B), \qquad q(B) \;:=\; I \;+\; \frac{B}{2!} \;+\; \cdots \;+\; \frac{B^{\mu-2}}{(\mu-1)!}\,.$$

Therefore since $B^\mu = 0$,

$$(e^B - I)^\mu \;=\; B^\mu q(B)^\mu \;=\; 0\,.$$

Therefore 1 is the only eigenvalue of e^B.

It follows that e^λ is the only eigenvalue of e^A. To prove this suppose that z is an eigenvalue of e^A and v an eigenvector so $e^A v = zv$. Then

$$e^B v \;=\; e^A e^{-\lambda I} v \;=\; e^{-\lambda} z v\,.$$

Thus $e^{-\lambda} z$ is an eigenvalue of e^B. Thus $e^{-\lambda} z = 1$, so, $z = e^\lambda$.

Step II. Write $X = \oplus X_j$ the spectral decomposition of A. Since the X_j are invariant under A one has

$$e^A \ = \ \oplus \, e^{A|_{X_j}} \,,$$

where $A|_{X_j} : X_j \to X_j$ denotes the restriction of A to X_j.

The result of Step I shows that $e^{A|_{X_j}}$ has only one eigenvalue, e^{λ_j}, with multiplicity equal to m_j. Theorem 6.1 follows. ■

Exercise 8.1 *The theorem is a spectral mapping theorem for* e^A*. To appreciate that this is a special case of a general phenomenon prove the following spectral mapping theorem for* A^n*. If* A *as in Theorem 8.1 and* n *is a positive integer, then the eigenvalues of* A^n *are the numbers* λ_j^n*. The multiplicity of* z *as an eigenvalue of* A^n *is the sum of the multiplicities of the* λ_j *such that* $\lambda_j^n = z$*.*