Kronecker's Theorem

Theorem 1. *If α is an irrational mulitple of 2π then the numbers*

$$e^{ik\alpha}\,, \qquad k \;=\; 0, 1, 2, \quad \cdots$$

are uniformly distributed on the circle S^1 in the sense that for any continuous function g on the circle,

$$\frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})\; d\theta \;=\; \lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N} g\big(e^{ik\alpha}\big)\,. \tag{1}$$

Proof. **I.** The proof concerns the linear functionals

$$g \;\mapsto\; \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})\; d\theta \;:=\; I(g),$$

and

$$g \;\mapsto\; \frac{1}{N+1}\sum_{k=0}^{N} g\big(e^{ik\alpha}\big) \;:=\; I_N(g)\,.$$

These are linear maps $C(S^1)\to\mathbb{C}$. On $C(S^1)$ use the norm

$$\|g\| \;:=\; \max_{\theta\in[0,2\pi]} |g(e^{i\theta})|\,.$$

One has the two elementary estimates,

$$|I(g)| \;\le \|g\|, \qquad \text{and} \qquad |I_N(g)| \;\le \|g\|\,. \tag{2}$$

II. The key step in the proof is to prove the assertion for

$$g(e^{i\theta}) \;=\; e^{in\theta}\,, \qquad n\in\mathbb{Z}\,.$$

For $n=0$, one has $g=1$, so $I(g)=I_N(g)=1$, proving the result.
For $n\ne 0$,

$$I(g) \;=\; \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\; d\theta \;=\; \frac{1}{2\pi}\frac{1}{in}\; e^{in\theta}\Big|_0^{2\pi} \;=\; 0\,.$$

Compute,

$$g(e^{ik\alpha}) \;=\; e^{ikn\alpha} \;=\; r^k, \qquad r := e^{in\alpha}\,.$$

The definition of I_N yields

$$I_N(g) \;=\; \frac{1}{N+1}\big(1 + r + r^2 + \cdots + r^N\big), \qquad r := e^{in\alpha}\,.$$

The hypothesis that α is not a rational multiple of 2π is equivalent to $r \neq 1$ for all $0 \neq n \in \mathbb{Z}$. Summing the series yields

$$|I_N(g)| \;=\; \frac{1}{N+1}\left|\frac{1 - r^{N+1}}{1-r}\right| \;\leq\; \frac{1}{N+1}\,\frac{2}{|1-r|} \;\to\; 0\,.$$

This verifies the result of the theorem for $e^{in\theta}$ and therefore for any finite linear combination,

$$G \;=\; \sum_{n=-\mu}^{\nu} a_n\, e^{in\theta}\,. \tag{3}$$

III. For an arbitrary $g \in C(S^1)$ and $\varepsilon > 0$, the Weierstrass Approximation Theorem asserts that there is a G as in (3) so that

$$\big\|g - G\big\| \;<\; \frac{\varepsilon}{3}\,.$$

Write

$$I(g) - I_N(g) \;=\; \big(I(g) - I(G)\big) \;+\; \big(I(G) - I_N(G)\big) \;+\; \big(I_N(G) - I_N(g)\big)\,. \tag{4}$$

The estmates (2) imply that

$$\big|I(g) - I(G)\big| \;=\; \big|I(g-G)\big| \;\leq\; \|g - G\| \;<\; \frac{\varepsilon}{3},$$

and,

$$\big|I_N(G) - I_N(g)\big| \;=\; \big|I_N(G-g)\big| \;\leq\; \|G - g\| \;<\; \frac{\varepsilon}{3}\,.$$

The result from **II** implies that the middle term on the right hand side of (4) tends to zero. So there is an N_0 so that for $N > N_0$,

$$\big|I(G) - I_N(G)\big| \;<\; \frac{\varepsilon}{3}\,.$$

Combining the estimates for the three terms in (4) shows that for $N > N_0$,

$$\big|I(g) - I_N(g)\big| \;<\; \frac{\varepsilon}{3} \;+\; \frac{\varepsilon}{3} \;+\; \frac{\varepsilon}{3} \;=\; \varepsilon\,,$$

which completes the proof. □