Math 558 Fall 2006 Prof. J.Rauch

Derivative of the Poincaré map December 19, 2006

It is sometimes possible to compute the derivatives of the Poincaré first return map at its fixed point, when the map itself is inaccessible. The procedure is described below. A simple example is given in Problem 3 of the 2006 Final Exam.

The system

$$X' = F(X)$$

has the T-periodic orbit $X(t)$.

Suppose that coordinates have been chosen so that

$$X(T) = X(0) = 0, \qquad X(t) \neq 0 \quad \text{for} \quad 0 < t < T\,,$$

and that

$$S = \{x_n = 0\}$$

is a section at 0.

Denote the flow by

$$\Phi(t, X) \;=\; (\phi_1(t, X), \ldots, \phi_n(t, X))\,.$$

The variational (a.k.a perturbation, a.k.a linearization) equation along the periodic orbit is

$$Y' = A(t)Y\,, \qquad A(t) = \partial_X F(X(t))\,.$$

$A(t)$ is an $N \times N$ matrix function of t. Denote by $Y(t)$ the solution whose initial value at $t = 0$ is the $N \times N$ identity matrix. Then

$$\partial_X \Phi(t, 0) = Y(t)\,, \tag{1}$$

Used when $t = T$ this is important for computing the Poincaré map. In addition one has

$$\partial_t \Phi(t, X) = F(X) \tag{2}$$

from the definition of flow. *If you compute $Y(T)$ you then know the first partial derivatives of $\Phi(t, X)$ at the important point $t = T, X = 0$.*

From these values one can compute the derivative of the Poincaré map by implicit differentiation. The time of first return $t(x_1, \ldots, x_{n-1}) = t(x^I)$ is given by

$$\phi_n(t(x^I), 0) = 0\,, \qquad t(0) = T\,. \tag{3}$$

With $x^I := (x_1, \ldots, x_n)$, the Poincaré map $P(x_1, \ldots, x_{n-1}) = P(x^I)$ is given by

$$P(x^I) = \Phi\big(t(x^I), (x^I, 0)\big)\,. \tag{4}$$

The derivative of P is computed by differentiating (4), and (3) with respect to the $n-1$ variables in x^I. Then set $x^I = 0$ using (1) and (2) for for the derivatives of Φ. (3) is one equation and (4) is $n - 1$ equations. Each has derivatives with respect to the $n - 1$ variabales x^I. This generates $n(n - 1)$ linear equations (with nonvanishing determinant) for the $n(n - 1)$ unknown derivatives of $t(x^I)$ and $P(x^I)$ at $x^I = 0$.