Zoom Zoom Zoom

1 Summary.

Repeatedly zooming in on a point $\underline{X}$ a vector field converges to a constant vector field with value $F(\underline{X})$. When $\underline{X}$ is an equilibrium the limiting vector field vanishes. If one rescales (plots on different axes) to compensate for the shrinking vectors, the limiting field becomes the linearization at $\underline{X}$. In this sense a vector field looks constant near a non equilibrium point and looks like the linearization near an equilibrium point.

2 Zooming

Suppose that

$$X' \;=\; F(X)$$

is a system of ordinary differential equations with continuously differentiable F. Want to examine the behavior of the vector field near a point $\underline{X} \in \mathbb{R}^N$. The first thing that we do is center the image on $\underline{X}$. Introduce

$$Y \;:=\; X - \underline{X}, \qquad \text{so} \qquad X \;=\; \underline{X} + Y\,.$$

The vector Y is the displacement from $\underline{X}$. The window with sides of length equal to 2 and center at $\underline{X}$,

$$\{X : |X_j - \underline{X}_j| < 1,\ 1 \le j \le N\}$$

becomes the window with side two centered at the origin in Y coordinates. The vector field in the Y coordinates is

$$F(\underline{X} + Y)\,.$$

Next zoom in on $Y = 0$ so that the window $|Y_j| < 1$ displays the values of the vector field that originally occupied the window $|Y_j| < 1/n$. The corresponding vector field is

$$F_n(Y) \;:=\; F(\underline{X} + Y/n)\,.$$

3 Zooms become constant

Proposition 3.1 *As $n \to \infty$, the sequence of vector fields F_n converge uniformly on the window $|Y_j| < 1$ to the the constant vector field $F(\underline{X})$.*

Proof. Since F is continuous at $\underline{X}$, given a challenge number $\varepsilon > 0$ choose δ_0 so that if $|X_i - \underline{X}_i| < \delta_0$ one has $\|F(X) - F(\underline{X})\| < \varepsilon$. Thus for $1/n < \delta_0$, and $|X_j| < 1$ one has

$$\|F_n(Y) - F(\underline{X})\| \;=\; \|F(\underline{X} + Y/n)) - F(\underline{X})\| \;<\; \delta\,.$$

$\blacksquare$

4 Normalized zooms at equilibria

At an equilibrium $\underline{X}$ the preceding proposition implies that the zoomed vector fields tend to zero. More precisely one has

$$F_n(Y) = F(\underline{X} + Y/n) \approx F(\underline{X}) + AY/n = AY/n, \qquad A := D_X F(\underline{X}).$$

Thus F_n is of size $\sim 1/n$ on the window $|Y_j| < 1$.

If one plots on a scale $\sim 1/n$ the zooms will not tend to zero in the limit. Define the *normalized zoom*

$$G_n(Y) := \frac{F_n(Y)}{1/n} = n\, F_n(Y). \tag{4.1}$$

Proposition 4.1 *If $\underline{X}$ is an equilibrium of the continuously differentiable vector field $F(X)$ then as $n \to \infty$ the normalized zooms $G_n(Y)$ converge uniformly on the window $|Y_j| < 1$ to the linearized field AY.*

Proof. Given a challenge number $\varepsilon > 0$ choose $\delta_0 > 0$ so that for $|X_i - \underline{X}_i| < \delta_0$

$$\big\| D_X F(X) - A \big\| = \big\| D_X F(X) - D_X F(\underline{X}) \big\| < \varepsilon / \sqrt{N}.$$

The Fundamental Theorem of Calculus reads

$$F(\underline{X} + Y/n) = \int_0^1 D_x F(\underline{X} + sY/n) ds\, Y/n.$$

Therefore

$$F_n(Y) - AY/n = \int_0^1 \Big(D_x F(\underline{X} + sY/n) - A \Big)) ds\, Y/n.$$

Thus for $1/n < \delta_0$ and $|Y_j| < 1$ one has

$$G_n(Y) - AY = \int_0^1 \Big(D_x F(\underline{X} + sY/n) - A \Big)) ds\, Y.$$

Since $\|Y\| \le \sqrt{N}$ one has,

$$\big\| G_n(Y) - AY \big\| < \frac{\varepsilon}{\sqrt{N}} \|Y\| \le \frac{\varepsilon}{\sqrt{N}} \sqrt{N} = \varepsilon$$

proving the proposition. ■

Exercise 4.1. *If $F(X) = BX$ is a linear system show that the normalized zoom of F at the equilibrium 0 is equal to the original vector field.*

Exercise 4.2. *Conversly, show that if 0 is an equilibrium of a continuously differentiable vector field F and the normalized zooms of F are equal to F, then F is linear.*

5 Long and very long times

Graphing the vector field on a small neighborhood of the equilibrium yields a vector field that closely resembles the linearization. It is not surprising that the linearization yields a good local approximation near the equilibrium. It is not hard to prove that the approximation is accurate on time intervals of length $\sim n$ on which the orbits move ~ 1 unit.

For times $\sim n^2$ the nonlinear terms in the Taylor expansion become important and the approximation by the linearization loses its precision.

Deeper is the fact that for linearizations that do not have purely imaginary eigenvalues the stable and unstable manifold structure that describes large time asymptotic behavior of the linear equation is inherited by the nonlinear equation.