Science Text Linearization

The basic computation is one that you will find in science texts very frequently. The system of differential equations is

$$X' \;=\; F(t,X) \qquad F = \big(f_1(t,X),\ldots,f_N(t,X)\big)\,.$$

One is interested in the behavior of solutions near a fixed solution $\underline{X}(t)$. One is supposed to have some knowledge of $\underline{X}$. Sometimes this background solution is very simple, for example an equilibrium.

Suppose that one considers a solution passing very close to $\underline{X}(t)$ at time $t = 0$. Write the exact solution as $X = \underline{X}(t) + \delta X(t)$ where $\delta X(t)$ is supposed to be small. The equations for $\underline{X}$ and X are then

$$\underline{X}' \;=\; F(t,\underline{X}), \qquad (\underline{X} + \delta X)' \;=\; F(t, \underline{X} + \delta X)\,.$$

Subtracting yields an equation exactly satisfied by the perturbation δX,

$$(\delta X)' \;=\; F(t, \underline{X} + \delta X) \;-\; F(t,\underline{X})\,.$$

So long as δX is small one has

$$F(t,\underline{X}(t) + \delta X(t)) \;-\; F(t,\underline{X}(t)) \;\approx\; D_X F(t,\underline{X}(t))\,\delta X(t)\,. \tag{1}$$

Here $D_X F$ denotes the Jacobian matrix

$$D_X F \;:=\; \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_N} \\ \frac{\partial f_2}{\partial X_1} & \cdots & \frac{\partial f_2}{\partial X_n} \\ & \cdots & \\ & \cdots & \\ & \cdots & \\ \frac{\partial f_N}{\partial X_1} & \cdots & \frac{\partial f_N}{\partial X_N} \end{pmatrix}$$

It is evaluated at $t, \underline{X}(t)$. Thus,

$$\delta X' \;\approx\; D_X F(t,\underline{X}(t))\,\delta X\,.$$

The perturbation is nearly a solution of the linear equation

$$U' \;=\; A(t)\,U\,, \qquad A(t) \;:=\; D_X F(t,\underline{X}(t))\,.$$

This equation is called the **linearization** or **variational equation**. The Linearization by Perturbation Theory handout gives a rigorous derivation.