Zeno's Paradoxes and Uniqueness

Abstract. This section discusses uniqueness. There is link between uniqueness and the Zeno paradoxes.

1 Rolling stops.

Consider a particle path in dimension one given by the equation $x(t) = t^3 + \epsilon t$ where $\epsilon > 0$ is very small. This motion moves right, slowing to a velocity ϵ at time $t = 0$ where it reaches $x = 0$.

If there were a stop sign at $x = 0$, then this motion never comes to stop. No matter how small is ϵ this motion deserves a ticket.

The case $\epsilon = 0$ is more subtle. The velocity hits zero for the unique instant of time $t = 0$. For any nonvanishing interval of time the car moves a finite distance forward. I suspect that the law requires a stop for a nonempty time interval but do not know how that would be expressed. I call such a motion a *rolling stop*.

1.1 Uniqueness implies no rolling stops.

Theorem 1.1 *Consider a differential equation*

$$x' = f(x) \tag{1.1}$$

with f continuously differentiable. A solution whose velocity is equal to zero for one instant of time must be an equilibrium.

Proof. Suppose that $x(t)$ is such a solution with $x'(\underline{t}) = 0$. Let $\underline{x} := x(\underline{t})$ be the position at time $\underline{t}$.

Then

$$0 \;=\; x'(\underline{t}) \;=\; f(x(\underline{t})) \;=\; f(\underline{x}) \,.$$

So $\underline{x}$ is an equilibrium.

The unique solution of (1.1) with $x(\underline{t}) = \underline{x}$ is then $x(t) = \underline{x}$. So $x(t)$ is an equilibrium. $\square$

1.2 Rolling stop with continuous f.

Somewhat surprisingly you can have a rolling stop and therefore do NOT have unique solutions of the initial value problem if f is merely assumed to be continuous. It is easy to verify[1] that $x(t) = t^3$ is a solution of

$$x' \;=\; 3\,|x|^{2/3}\,.$$

There are many other solutions, for example for any $\tau > 0$ the function equal to 0 for $t \leq \tau$ and equal to $(t-\tau)^3$ for $t > \tau$ is another solution.

Similarly, there are solutions which move right, come to a rest at the origin and take off at a positive time.

Exercise. *Verify.*

2 Zeno paradox.

One classical paradox of Zeno meant to show the danger of imprecise thinking and in particular the imprecision of language demonstrates that a frog hopping at constant speed in the direction of a pond will never reach the pond.

The reasoning is the following. Before the frog reaches the pond it reaches after a time ΔT_1 the point which is half way to the pond. Call this event 1.

Before reaching the pond the frog must then reach after an additional time ΔT_2 the point half way again to the pond. This is event 2.

In this way, before the frog reaches the pond an infinite number of events separated by positive times $\Delta T_j > 0$ must occur.

This infinite number of nonvanishing time intervals must take infinitely long so the frog does not reach the pond in finite time.

Continuing, you can apply the same argument to show that the frog never gets to half way to the pond. Continuing you find that the frog cannot get anywhere.

2.1 Time does not move either.

The sentence in italics sounds reasonable but is very wrong.

[1]This is a simple modification of the example on page 384 of Hirsh-Smale-Devaney. You can find these solutions in $x \neq 0$ by separation of variables.

You can see this by watching the passage of time.

Let T be the time it should take the frog to get to the pond. Event 1 occurs after $T_1 = T/2$ units of time.

Event 2 occurs after an interval of time half again as long. And so on. Before T units of time have elapsed an infinite sequence of positive time intervals must pass. The sentence in italics asserts this is impossible so T units of time cannot pass.

Continuing one conclues that time cannot progress either.

2.2 The error.

Brought up as we are on limits and infinite series we know that $\Delta T_j = \Delta T_1/2^j$ are summable,

$$\sum_{j=0}^{\infty} \frac{\Delta T_1}{2^j} \;=\; 2\,\Delta T_1 \;<\; \infty\,.$$

That

$$\sum_{j=1}^{\infty} \frac{1}{2^j} \;=\; \frac{1}{2} \;+\; \frac{1}{4} \;+\; \frac{1}{8} \;+\; \cdots \;=\; 1$$

is easily read from the markings between 0 and 1 on a ruler.

3 The ode computation.

Consider the Fundamental Theorem of the Phase Line with f continuously differentiable, positive on $]a,b[$ and vanishing on $[a,b]$. The fact that there are no rolling stops asserts that solutions with $a < x(0) < b$ takes an infinite amount of time to reach b.

Consider the intervals $I_n := [b - 2^{-n}, b - 2^{n+1}]$.

We know that $f(b) = 0$. Let

$$M \;:=\; \max_{x\in[a,b]} |f'(x)|\,.$$

Then for $x \in [a,b]$

$$f(x) \;=\; f(x) - f(b) \;=\; -\int_x^b f'(x)\;dx \;\le\; \int_x^b M\;dx \;=\; M\,(b-x)\,.$$

Therefore

$$x \in I_n \;\Rightarrow\; x' \;=\; f(x) \;<\; \frac{M}{2^n}\,.$$

Thus when the solution lies in I_n its velocity is at most $M/2^n$. To cross I_n takes time at least

$$\frac{|I_n|}{M/2^n} \;=\; \frac{1/2^{n+1}}{M/2^n} \;=\; \frac{1}{2M}$$

Now the Zeno argument works since passing each intervals I_n takes at least $1/2M$ units of time. These add to an infinite amount of time.

Exercise. *Consider a rolling stop solution in $x < 0$ from §1.2. Compute that for the ordinary differential equation in that section the time taken to cross the analogous I_n sum to a finite quantity.* **Hint.** Show that when $x \in I_n$, $3/2^{2(n+1)/3} < x'$.