Linearization Unstable but Nonlinear Equilibrium Stable

Summary. *A basic instability theorem asserts that if $X' = F(X)$ has an equibilibrium $\underline{X}$ and the linearization at $\underline{X}$ has an eigenvalue with strictly positive real part, then the equilibrium is unstable. The more general assertion "linearization unstable implies equilibrium unstable" is false. We give a very simple example.*

The point of departure is the linear spring

$$\ddot{x} \;+\; k^2\, x \;=\; 0, \qquad k > 0\,.$$

This linear problem has the pair of complex conjugate purely imaginary eigenvalues $\pm ik$. It is stable. The stability can also be proved using the the conserved energy

$$E \;:=\; \frac{\dot{x}^2}{2} + k^2\, \frac{x^2}{2}\,.$$

The latter proof extends to more general attractive springs. For example

$$\ddot{x} \;+\; x^3 \;=\; 0\,,$$

has the conserved energy

$$E \;:=\; \frac{\dot{x}^2}{2} + \frac{x^4}{4}\,.$$

Therefore, the equilibrium and also the equilibrium $(0,0)$ of the system version

$$\dot{x}_1 \;=\; x_2\,, \qquad \dot{x}_2 \;=\; -x_1^3$$

are stable.

The linearization at the equilibrium is

$$\ddot{x} = 0$$

or the system version

$$\dot{X} \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} X\,.$$

They are unstable with solutions that grow linearly in time.

The linearization is unstable. The nonlinear equilibrium is stable.

Remarks. 1. The instability theorem asserts that this could not happen if the linearization has a solution that grows exponentially.

2. This example complements the standard examples with linearizations that are stable centers and nonlinear dynamics unstable. For example

$$\dot{X} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X \;+\; \varepsilon \|X\|^2 X\,, \qquad \varepsilon > 0\,.$$

Exercise 1 *Denote the velocity $v := x'$. The orbits are then the energy level sets*

$$\Big\{(x,v) \;:\; \frac{x^2}{2} \;+\; \frac{v^4}{4} \;=\; E\Big\}\,.$$

Show that the level sets have the following properties.

i. *They are symmetric under reflection is both the x and v axes.*

ii. *They are perpendicular to both the x and v axes.*

iii. *The intercepts with the x-axis are*

$$\big(\pm (2E)^{1/2}, 0 \big) .$$

The intercepts with the v-axis are

$$\big(\pm (4E)^{1/4}, 0 \big) .$$

Discussion. The orbits near the origin correspond to $0 < E << 1$. For these the v intercepts are much larger than the x intercepts. The ovaloid orbits are very long and very narrow. The long direction is along the v-axis.

Exercise 2 *Denote by Γ the level set of energy $E = 1$. Show how knowing Γ determines all the other level sets.*

Exercise 3 *Show that the level sets have strictly positive curvature. Equivalently they bound a strictly convex set.*