Asymptotic Stability by Linearization

Summary. *Sufficient and nearly sharp sufficient conditions for asymptotic stability of equiiibria of differential equations, fixed points of maps, and periodic orbits of differential equations can all be given in terms of spectral information of linearized problems. The common ingredient is the existence of scalar products whose norm decreases for the linearization.*

1 Asymptotic stability of equilibria

The main result asserts that if the linearization at an equilibrium is asymptotically stable, then the equilibrium is asymptotically stable.

The theorem in §6 of the Spectral Theorem handout shows that *if A is a linear transformation of a finite dimensional complex vector space $\mathbb{V}$ and the eigenvalues of A have strictly negative real part, then there is a scalar product $Q(X,Y)$ on $\mathbb{V}$ and an $\alpha < 0$, so that solutions of $X' = AX$ satisfy*

$$\frac{d}{dt} Q(X(t), X(t)) \ \leq \ \alpha\, Q(X(t), X(t))\,.$$

Suppose that $F(X)$ is a twice continuously differentiable function on a neighborhood of $\underline{X}$ and $F(\underline{X}) = 0$. Study the stability of $\underline{X}$ as an equilibrium of the differential equation

$$X' \ = \ F(X)\,. \tag{1.1}$$

The linearization at the equilibrium is

$$Y' \ = \ AY\,, \qquad \text{with} \qquad A := D_X F(\underline{X})\,.$$

It is asymptotically stable if and only if the eigenvalues of A have strictly negative real part.

Theorem 1.1 *If the eigenvalues A have strictly negative real part then the equilibrium $\underline{X}$ is asymptotically stable for the nonlinear dynamics* (1.1)

Remark 1.1 *A modification of the proof shows that the result is also true if F is assumed to be only one time continuously differentiable.*

Proof. Step I. Taylor expansion. Denote by $Z := X - \underline{X}$ the displacement from the equilibrium. The equation for Z is found from

$$X' \;=\; (\underline{X} + Z)' \;=\; F(X) \;=\; F(\underline{X} + Z)\,.$$

Therefore

$$Z' = F(\underline{X} + Z)\,.$$

Denote by Q a scalar product as in §1. From here on measure all distances using Q. Choose $r_1 > 0$ so small that the closed ball (defined with the distance Q) with radius $r_1 > 0$ and center $\underline{X}$ belongs to the domain where F is twice differentiable. Taylor's theorem with remainder implies that there is a constant C so that for $\|Z\| \le r_1$,

$$F(\underline{X} + Z) \;=\; F(\underline{X}) + AZ + R(Z)\,, \qquad \text{with} \qquad \|R(Z)\| \le C\|Z\|^2\,.$$

Step II. The fundamental "so long as" estimate. On solutions of the linearized equation, $Y' = AY$ one has $Q(Y,Y)' \le \alpha Q(Y,Y)$. Evaluating the left hand side yields

$$Q(Y,Y)' \;=\; Q(Y',Y) + Q(Y,Y') \;=\; Q(AY,Y) + Q(Y,AY)\,.$$

Therefore

$$Q(AY,Y) + Q(Y,AY) \;\le\; \alpha Q(Y,Y)\,. \tag{1.2}$$

So long as $Q(Z,Z) \le r_1^2$ compute the time derivative of $Q(Z,Z)$

$$\begin{aligned}
Q(Z,Z)' \;&=\; Q(Z',Z) + Q(Z,Z') \\
&=\; Q(AZ + R(Z)\,,\, Z) \;+\; Q(Z\,,\, AZ + R(Z)) \\
&=\; \Big(Q(AZ\,,\, Z) \;+\; Q(Z, AZ)\Big) \;+\; \Big(Q(R(Z), Z) \;+\; Q(Z, R(Z))\Big) \\
&:=\; I_1 \;+ I_2\,.
\end{aligned} \tag{1.3}$$

Estimate (1.2) implies that $I_1 \le \alpha Q(Z,Z)$. For I_2 use the Cauchy-Schwartz inequality for the scalar product Q to estimate

$$\begin{aligned}
I_2 \;&=\; 2\,\mathrm{Re}\, Q(R(Z),Z) \;\le\; 2\,|Q(R(Z),Z)| \\
&\le\; 2\, \|R(Z)\|\||Z\| \;\le\; 2\,C\,\|Z\|^2\,\|Z\| \;=\; 2\,C\,Q(Z,Z)^{3/2}\,.
\end{aligned}$$

Using these two estimates in (1.3) yields

$$Q(Z,Z)' \;\le\; \Big(\alpha + 2\,C\,Q(Z,Z)^{1/2}\Big)Q(Z,Z)\,.$$

Choose $r_2 \in]0, r_1]$ *so that* $2\,C\,r_2 \ \leq\ \frac{|\alpha|}{2}$. *Then so long as* $Q(Z,Z) \leq r_2$ *one has*

$$\Big(\alpha + 2\,C\,Q(Z,Z)^{1/2}\Big)\ \leq\ \frac{\alpha}{2}\,, \qquad \text{so,} \qquad Q(Z,Z)'\ \leq\ \frac{\alpha}{2}\,Q(Z,Z)\,. \quad (1.4)$$

Step III. The balls $Q(Z,Z) \leq r^2$ **are invariant for** $r \leq r_2$**.** The case $r = 0$ is trivial. Suppose that $0 < r \leq r_2$ and that $Q(Z(0), Z(0)) = r^2$. Must show that $Q(Z,Z) \leq r^2$ for all $t \geq 0$.

Since at $t = 0$, $Q(Z(0), Z(0)) = r^2 \leq r_1^2$, one has $Q' \leq (\alpha/2)Q(Z,Z) < 0$ so there is a $\delta > 0$ so that $Q(Z(t), Z(t)) < r^2$ for $0 < t < \delta$.

If it were not true that $Q(Z(t), Z(t)) < r^2$ for all $t > 0$, there would be a smallest $T > 0$ so that $Q(Z(T), Z(T)) = r^2$. Then for $0 < t < T$ one has $0 < Q(Z(t), Z(t)) < r^2$ so $Q' < 0$. Then there is a contradiction from

$$0 = r^2 - r^2 = Q(Z(T), Z(T)) - Q(Z(0), Z(0))\ =\ \int_0^T \frac{d}{dt}\,Q(Z(t), Z(t))\,dt\ <\ 0\,.$$

The invariance of the Q-balls implies stability.

Step IV. Asymptotic stability. If $Q(Z(0), Z(0)) \leq r_2^2$ then $Q(Z(t), Z(t)) \leq r_2^2$ for all $t \geq 0$ and the second inequality in (1.4) holds so

$$\Big(e^{-\alpha t/2}\ Q\Big)'\ =\ e^{-\alpha t/2}\Big(Q' - \frac{\alpha}{2}\,Q\Big)\ \leq\ 0\,,$$

so for all $t > 0$

$$e^{\alpha t/2} Q(Z(t), Z(t))\ \leq\ Q(Z(0), Z(0))$$

proving that $Z(t)$ decays exponentially to zero. In particular the equilibrium is asymptotically stable. □

2 Asymptotic stability of fixed points

The linearizaiton sufficient condition for asymptotic stability of a fixed point is the following.

Theorem 2.1 *Suppose that* $\mathbb{V}$ *is finite dimensional,* $P : \mathbb{V} \to \mathbb{V}$ *is continuously differentiable and* $\underline{q}$ *is a fixed point of* P*. If all the eigenvalues of the derivative* $DP|_{\underline{q}}$ *have modulus strictly less than one, then* $\underline{q}$ *is asymptotically stable.*

Proof. The *Spectral Theory Handout* shows that there is a scalar product Q and $0 < a < 1$ so that

$$\|DP|_{\underline{q}}(\sigma)\|_Q \ < \ a\|\sigma\|_Q \qquad \text{for all} \ \ \sigma\,.$$

It follows that the derivative satisfies the same inequality for all q sufficiently close to $\underline{q}$. Therefore if one chooses $r > 0$ sufficiently small one has for $q \in S$

$$\|q - \underline{q}\|_Q \ \le \ r \quad \Longrightarrow \quad \|DP|_q(\sigma)\|_Q \ < a\,\|\sigma\|_Q\,. \tag{2.1}$$

For q in the ball $\|q - \underline{q}\|_Q \ \le \ r$, write $\|P(q) - \underline{q}\|_Q \ = \ \|P(q) - P(\underline{q})\|_Q$. Express the latter as the integral over $0 \le s \le 1$ of

$$\frac{d}{ds}\ P(\underline{q} + s(q - \underline{q})) \ = \ DP|_{\underline{q}+s(q-\underline{q})}(q - \underline{q})\,.$$

Estimate using (2.1)

$$\|DP|_{\underline{q}+s(q-\underline{q})}(q - \underline{q})\|_Q \ < \ a\,\|q - \underline{q}\|_Q$$

to find

$$\|P(q) - \underline{q}\|_Q \ < \ \int_0^1 a\,\|q - \underline{q}\|_Q\ ds \ = \ a\,\|q - \underline{q}\|_Q\,.$$

By induction on n,

$$\|P^n(q) - \underline{q}\|_Q \ < \ a^n\,\|q - \underline{q}\|_Q\,,$$

proving asymptotic stability of the fixed point follows. □

Exercise 2.1 *Give an alternate proof Theorem 1.1 by applying Theorem 2.2 to the nonlinear map $X \mapsto \Phi_1(X)$ with fixed point equal to $\underline{X}$.* **Hint.** Use the Spectral Mapping Theorem at the end of the Spectral Theory handout

3 Orbital asymptotic stability of periodic orbits

Consider the autonomous system

$$X' \ = \ F(X)\,. \tag{3.1}$$

Suppose that $\underline{X}(t)$ is a periodic orbit with period $T > 0$ with linearization

$$Y' \ = \ A(t)\,Y\,, \qquad A(t) \ := \ D_X F(\underline{X}(t))\,. \tag{3.2}$$

This section proves the sufficient condition for asymptotic stability from the linearization. The 2×2 case is treated in HSD §10.3.

Since $\delta \mapsto X(t+\delta)$ is a smooth one parameter family of solutions of $X' = F(X)$ its derivative

$$Y(t) \ := \ \frac{d}{d\delta}\, X(t+\delta)\big|_{\delta=0} \ = \ X'(t)$$

is a solution of the linearized equation (3.2) and it is T-periodic. Therefore

$$\Psi(T)X'(0) \ = \ \Psi(T)Y(0) \ = \ Y(T) \ = \ Y(0) \ = \ X'(0)$$

showing that $X'(0)$ is an eigenvector of $\Psi(T)$ with eigenvalue equal to 1.

Theorem 3.1 *If the the fundamental matrix $\Psi(T)$ of the linearization at the T-periodic orbit $\underline{X}(t)$ with $\Psi(0) = I$ has eigenvalues equal to 1 and $N-1$ eigenvalues of modulus less than one, then the periodic orbit $\underline{X}(t)$ is orbitally asymptotically stable.*

The proof proceeds by analysing the first return map. Let $\underline{q} := X(0)$. Translating coordinates suppose that $\underline{q} = 0$. Define $\mathbf{v} := X'(0) = F(0)$. Choose a linear subspace $\mathbb{W}$ complementary to the subspace $\mathbb{R}\,\mathbf{v}$,

$$\mathbb{V} \ = \ \mathbb{R}\,\mathbf{v} \ \oplus \ \mathbb{W}\,.$$

Introduce the Poincaré first return map $P : \mathcal{O} \to \mathbb{W}$ defined on a neighborhood $\mathcal{O}$ of $\underline{q}$ in $\mathbb{W}$. The eigenvalues of the derivative of the first return map are the following.

Theorem 3.2 *Denote by $\Psi(t)$ the fundamental matrix,*

$$\Psi'(t) \ = \ A(t)\,\Psi(t)\,, \qquad \Psi(0) = I\,.$$

Then 1 is an eigenvalue of $\Psi(T)$ with eigenvector equal to $\mathbf{v}$. The other $N-1$ eigenvalues of $\Psi(T)$ are the $N-1$ eigenvalues repeated according to multiplicity of $DP|_{\underline{q}}$.

Proof of Theorem 3.2. Step I. Analyse the first return time. Denote by Φ_t the flow of the equation $X' = F(X)$. For $w \in \mathbb{W}$ the first return time is defined by

$$\Phi_{t(w)}(w) \ \in \ \mathbb{W}\,, \qquad t(\underline{q}) \ = \ T\,.$$

To analyze choose a nontrivial linear $\ell : \mathbb{V} \to \mathbb{R}$ so that $\mathbb{W} \ = \ \ker \ell$. Define

$$f(t,w) \ = \ \ell\,\Phi_t(w)$$

for w on a neighborhood of 0 in $\mathbb{W}$. The equation defining $t(w)$ is then

$$f(t(w)\,,\,w) \;=\; 0\,, \qquad t(\underline{q}) \;=\; T\,. \tag{3.3}$$

The implicit function Theorem yields a unique differentiable local solution provided that $f_t(T,\underline{q}) \neq 0$.
Compute

$$\frac{\partial f}{\partial t} \;=\; \frac{\partial}{\partial t}\,\ell\big(\Phi_t(w)\big) \;=\; \ell\bigg(\frac{\partial}{\partial t}\,\Phi_t(w)\bigg) \;=\; \ell\big(F(\Phi_t(w))\big) \;=\; \ell(\mathbf{v}) \;\neq\; 0$$

since $\mathbf{v} \notin \mathbb{W}$.

Step II. Express the first return map as

$$P(w) \;=\; \Phi_{t(w}(w)\,.$$

Differentiating with respect to w yields

$$D_w P \;=\; \frac{\partial \Phi}{\partial t}\bigg|_{t=t(w)} D_w t(w) \;+\; D_w \Phi(t,w)\big|_{t=t(w)}$$

Evaluate at $w=\underline{q}$ to find

$$D_w P(\underline{q}) \;=\; \mathbf{v}\; D_w t(w) \;+\; \Psi(T)\,. \tag{3.4}$$

Introduce a basis for $\mathbb{V}$ whose first element is $\mathbf{v}$ and whose last $N-1$ elements form a basis for $\mathbb{W}$. Then equation (3.4) together with $\psi(T)v = v$ imply that the matrix for $\Psi(T)$ in this basis is of the form

$$\Psi(T) \;=\; \begin{pmatrix} 1 & * & \dots & * \\ 0 & & & \\ 0 & & D_w P(\underline{q}) & \\ 0 & & & \end{pmatrix}.$$

The conclusion of the Theorem follows. $\square$

Proof of Theorem 3.1. The hypothesis of the Theorem 3.1 together with Theorem 3.2 imply that $DP(\underline{q})$ has only eigenvalues of modulus strictly less than one. Theorem 2.1 implies that $\underline{q}$ is an asymptotically stable fixed point of the first return map P. This implies the orbital asymptotic stablility of the periodic orbit. $\square$