Dynamics in Dimension 1.5

1 Introduction.

Supplements to section 1.4 and 1.5 of [1]. Those sections consider ordinary differential equation of the form

$$x' \;=\; f(t,x) \tag{1.1}$$

where $x(t)$ is real valued and $f(t,x)$ is a continuously differentiable function of t, x. The unknown is scalar but the problem is **nonautonomous** when f depends on t. The dynamics can be more complicated than the autonomous scalar case but not as complicated as autonomous planar systems. It is for that reason that it is called dimension 1.5.

After autonomous planar systems, the next level of complexity is non autonomous planar systems which are called dimension 2.5. It is there that the first chaotic examples appear.

Important ideas introduced are the concepts of the flow, monotone map, and Poincaré map. The method of integrating factors for scalar linear equations, and Perturbation Theory are introduced to analyse an example. A thorough treatment of dynamics in $d = 1.5$ is Part II of [2].

2 The flow and Poincaré map.

Section 1.4 introduces the **flow** $\phi(t, x_0)$ which is the value at time t of the solution $x(t)$ with $x(0) = x_0$. The fact that $t \mapsto \phi(t, x_0)$ satisfies the differential equation says that

$$\frac{\partial \phi}{\partial t} \;=\; f\big(t\,,\,\phi(t,x_0)\big), \qquad \phi(0,x_0) \;=\; x_0\,. \tag{2.1}$$

The fundamental existence and uniuqeness theorem implies that ϕ is a continuously differentiable function defined on an open [1] subset of $\mathbb{R}_t \times \mathbb{R}_{x_0}$. It may not be everywhere defined since solutions may diverge to infinity in finite time.

[1] A subset $\Omega \subset \mathbb{R}^d$ is **open** when for each point $y \in \Omega$ there is an $r > 0$ so that the ball $B_r(y) := \{z : |z - y| < r\}$ is a subset of Ω.

If one fixes t then the map

$$x_0 \ \mapsto \ \phi(t, x_0)$$

maps the initial datum to the value of the solution at time t. It is a map from the x-line to itself describing the dynamics of the differential equation between time $t = 0$ and t. [2]

More generally $\phi(t, t_0, x_0)$ is the value at time t of the solution satisfying $x(t_0) = x_0$. One can continue the solution starting at $t = 0$ by taking $\phi(t, 0, x_0)$ for $0 \le t \le t_1$. Then $\phi(t, t_1, \phi(t_1, 0, x_0))$ for $t \ge t_1$. This corresponds to taking initial datum $\phi(t_1, 0, x_0)$ at time t_1. And of course one can repeat this over and over.

This becomes particularly interesting *when f is periodic in t with period equal to 1.* In that case if $x(t)$ is a solution then so is $x(t-1)$ whose graph is that of x translated one unit to the right. It remains tangent to the slope field of the differential equation $x' = f(t, x)$ because that field is invariant under translation to the right by one unit. The same is true for translation to the left or right by any integer $n \in \mathbb{N}$.

In this periodic case define a map denote by $p(x_0) := \phi(1, x_0)$. It is called the **Poincaré map**. Then $p(x)$ gives the position at time $t = 1$ of the solution starting at $t = 0$ at the point x. If one knows $p(x_0)$, the values in the next period $1 \le t \le 2$ are found as the solution of the initial value problem

$$x' = f(t, x), \qquad x(1) = p(x_0)\,.$$

This is the translation to the right by one unit of the the solution $\underline{x}(t)$ of

$$\underline{x}' = f(t, \underline{x}), \qquad \underline{x}(0) = p(x_0)\,.$$

Therefore the solution at time 2 is equal to

$$x(2) \ = \ \underline{x}(1) \ = \ \phi(1, 0, p(x_0)) \ = \ p\big(p(x_0)\big) \ = \ (p \circ p)(x_0)\,.$$

The iterates

$$p, \quad p^2 := p \circ p, \quad p^3 := p \circ p \circ p, \quad \ldots$$

describe the behavior after one, two three periods, by

$$x(n) \ = \ p^n(x_0)\,.$$

[2] If some solutions diverge to infinity at times $\le t$ then $\phi(t, x)$ will be defined only on a subset of $x \in \mathbb{R}$.

Warning: This is not the n^{th} power but the n^{th} composition. The sequence $p^n(x)$ is called the (future) orbit of x. Be sure that you understand why this depends essentially on the periodicity of f.

When x belongs to the range of p the uniqueness theorem implies that it has only one preimage. Denote by $p^{-1}(x)$ its preimage . Similarly

$$p^{-2} := p^{-1} \circ p^{-1}, \qquad p^{-3} := p^{-1} \circ p^{-1} \circ p^{-1}, \quad etc.$$

The sequence $p^{-n}(x)$ is called the past orbit of x for as long as it remains well defined. The domain of p^{-n} consists of points that have n generations of predecessors. The domain is non increasing in n.

Example. For the logistic equation with harvesting there is an $x_* \in \mathbb{R}$ so that solutions with initial value larger than x_* exist throughout $0 \le t \le 1$ while the solution with initial value x_* diverges to $-\infty$ as t increases to 1. The domain of p is $]x_*, \infty[$ and p maps this interval one to one and onto $]-\infty, \infty[$. The maps p^{-n} are everywhere defined for $n > 0$ and the domains of p^n decrease with n.

Fixed points of p correspond to solutions of the ordinary differential equation that are periodic with period 1. The behavior of solutions of an ordinary differential equation has been replaced by the behavior of the iterates of a map. The study of iterates is the subject of **Dynamical Systems** and is intimately related to differential equations. This close relation was brought to prominence in the work of Poincaré.

The example in §1.5 supplemented with these notes shows how it can be used to analyse the logistic equation with periodic harvesting.

3 Monotonicity.

The Poincaré map when x is one dimensional has an important monotonicity property, even when the problem is not periodic.

Definition 3.1 *A map p from an interval $I \subset \mathbb{R}$ to $\mathbb{R}$ is called strictly monotone when $x_1 < x_2 \Rightarrow p(x_1) < p(x_2)$.*

Consider two solution curves $(t, x(t))$ and $(t, \widetilde{x}(t))$ the first starting below the second, $x_0 < \widetilde{x}_0$. The upper orbit must stay above because orbits cannot cross.

Theorem 3.1 *Suppose that* $x_0 < \widetilde{x}_0$ *and that* $x(t)$ *and* $\widetilde{x}(t)$ *are the solutions with those initial data. If both* x *and* $\widetilde{x}$ *exist for* $0 \le t \le T$ *then for those* t*,*

$$x(t) \ < \widetilde{x}(t)\,.$$

Proof. Define $d(t) = \widetilde{x}(t) - x(t)$. The theorem asserts that $d > 0$. To show that the continuous function d is strictly positive it suffices (by the Intermediate Value Theorem for continuous functions) to show that d can never vanish.

If there were a $0 < \underline{t} \le T$ with $d(\underline{t}) = 0$ then $x(t)$ and $\widetilde{x}(t)$ would be solutions of (1.1) with the same value when $t = \underline{t}$. The uniqueness theorem implies that they must be equal throughout. In particular

$$0 \ = \ d(0) \ = \ \widetilde{x}(0) \ - \ x(0) \ > \ 0\,.$$

This contradiction shows that $d(\underline{t}) = 0$ is impossible so $d < 0$. □

This shows that the Poincaré map is strictly monotone. This implies that the dynamics cannot be very complicated. There is no chaos in 1.5 dimensional systems.

4 Fundamental theorem of monotone maps.

The next result is closely related to the Fundamental Theorem of the Phase Line.

Theorem 4.1 *Suppose that* $p : [a, b] \to \mathbb{R}$ *is a continuous and strictly monotone map with fixed points* a *and* b *and no other fixed points in* $]a, b[$*. Then*

- p *maps the interval* $[a, b]$ *one to one and onto itself.*
- $p(x) - x$ *has one sign on* $]a, b[$*.*
- *If* $p(x) - x > 0$ *(resp.* < 0*) on* $]a, b[$ *then for any* $x \in]a, b[$ *the orbit* $x, p(x), p^2(x), \ldots$ *converges to* b *(resp.* a*).*
- *If* $p(x) - x > 0$ *(resp.* < 0*) then for any* $x \in]a, b[$ *the past orbit* $x, p^{-1}(x), p^{-2}(x), \ldots$ *converges to* a *(resp.* b*).*

Proof. If $a < x < b$ then monotonicity implies $a = p(a) < p(x) < p(b) = b$ proving that p maps $[a, b]$ to itself.

If $a \le x_1 < x_2 \le b$ then monotonicity yields $p(x_1) < p(x_2)$ showing that p is one to one.

Since $p(a) = a$ and $p(b) = b$, the intermediate value property of continuous functions implies that for any $y \in [a, b]$ there is an $x \in [a, b]$ so that $p(x) = y$ proving that the map is onto.

Monotonicity and $p(x) > x$ imply that for $x \in]a, b[$, the sequence $p^n(x) < b$ is strictly increasing. An increasing sequence bounded above has a limit

$$\lim_{n\to\infty} p^n(x) \;=\; \overline{x} \;\leq\; b.$$

It remains to prove that $\overline{x} = b$.

Using continuity for the second equality we have,

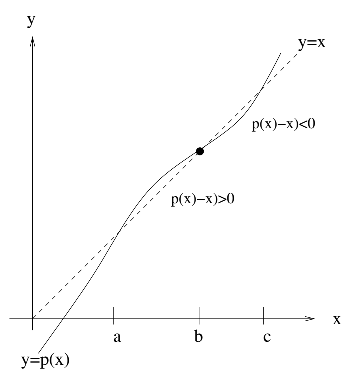
$$p(\overline{x}) \;=\; p\big(\lim p^n(x)\big) \;=\; \lim p\big(p^n(x)\big) \;=\; \lim p^{n+1}(x) \;=\; \overline{x}\,.$$

Thus $\overline{x}$ is a fixed point with $a < \overline{x} \leq b$. By hypothesis, b is the only such fixed point, so $\overline{x} = b$. □

5 Long term behavior.

When f is t-periodic, each root of $p(x) - x$ is the initial value of an orbit of $x' = f(t, x)$ that is periodic with period equal to 1. The fundamental theorem shows that solutions starting in an interval between roots converge to one or the other of the bounding periodic orbits. The long term behavior of all solutions is determined by the roots of $p(x) - x$ and its sign in between.

Example. Suppose that the Poincaré map of a periodic differential equation is as in the figure.



The orbits starting at the fixed points a, b, and c of p are periodic with period equal to 1. The orbit through b lies above that through a and below that through c. All orbits with initial values $a < x_0 < b$ converge to the periodic orbit through b.

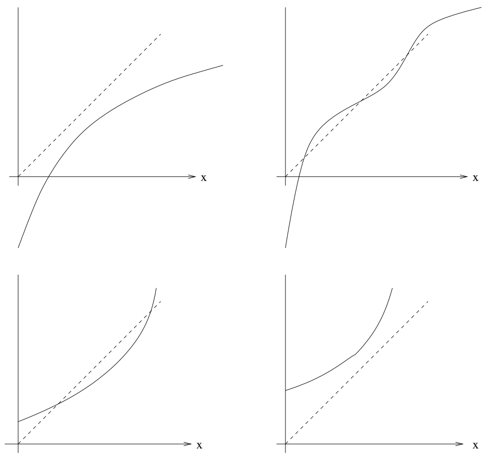
Exercise 5.1 *Draw a diagram analogous to Figure 1.11 of Hirsch-Smale-Devaney illustrating this situation.*

Exercise 5.2 *Suppose that $p(x)$ is a monotone map with only two fixed points $x_1 < x_2$. They divides the real line into three intervals and on each the sign of $p(x) - x$ is constant. Suppose that $p(x) - x$ is positive on $-\infty < x < x_1$. There remain four possibilites for the signs in the remaining two intervals yielding sign patterns $+, -, -$, $+, +, -$, $+, -, +$, and $+, +, +$. For each of the four possibilities find the future limit and past limit of the orbits of points starting in each of the three intervals.*

Algorithm. *The sign pattern of $p(x) - x$ determines the dynamics of a periodic $x' = f(t, x)$ just as the sign pattern of $f(x)$ detemines the dynamics of $x' = f(x)$. That sign pattern is found by either plotting $p(x)-x$ or plotting both $p(x)$ and x on the same graph.*

To determine the sign patterns requires the determination of the roots of $p(x)-x$. In almost all cases that can only be done approximately. When the roots are not very close together and curves cross the x-axis with slope not too small this can be done with accuracy and confidence. One strategy that gives additional information is to compute the values of $p(x)-x$ at regularly spaced points. Even when p is given by an explicit formula, for example $\cos x$, its values can only be computed approximately. The values of $p(x)$ can be computed with comparable accuracy and comparable low computational complexity using a numerical method for approximately solving ordinary differential equations.

Exercise 5.3 *Each of the four figures sketches a Poincaré map $p(x)$ on the same axes as the dotted line which is a graph of x. For each graph determine the fixed points of p. They correspond to periodic orbits of the differential equation. Determine the long time behavior of solutions (both past and future) of the differential equation for initial data starting in each of the intervals bounded by equilibria or $\pm\infty$. You may assume that the solutions of the ordinary differential equation do not blow up in finite time.*



6 Derivative test for stability of periodic orbits.

There are derivative criteria involving $f'(\underline{x})$ for stability of equilibria in the autonomous case. In the periodic case there are analogous criteria involving $p'(\underline{x})$ for the stability of periodic orbits.

Equilibria $\underline{x}$ of $x' = f(x)$ are stable (resp. unstable) when $f'(\underline{x}) < 0$ (resp. > 0).

A first criterion that generalizes easily to higher dimensional problems is that a fixed point $\underline{x}$ of $p(x)$ yields a stable (resp. unstable) periodic orbit when $|p'(\underline{x})| < 1$ (resp. > 1). This is reasonable since for $x \approx \underline{x}$,

$$|p(x) - \underline{x}| = |p(x) - p(\underline{x})| \approx \left|p'(\underline{x})(x - \underline{x})\right| = |p'(\underline{x})|\,|x - \underline{x}|\,.$$

When $|p'(\underline{x})| < 1$ and $x \approx \underline{x}$, then $p(x)$ will be closer to $\underline{x}$ than x was.

More generally, an isolated equilibrium, $\underline{x}$ of $p(x) - x$ is stable $p(x) - x$ is positive to the left of $\underline{x}$ and negative to the right. A sufficient condition is that $p(x) - x$ has negative derivative at $\underline{x}$. This holds if and only if $p'(x) < 1$, a sharper criterion than $|p'(\underline{x})| < 1$.

7 Convcavity of $p(x)$.

This material presents an alternative version of page 13 of [1]. It is better because it introduces **Perturbation Theory** that is useful in other contexts too.[3] The goal is to compute the derivatives of the Poincaré map $p(x)$. This amounts to studying what happens to a solution of the differential equation when the initial condition is changed a little bit. The first result is a strengthening of monotonicity. The computation is an application of the Fundamental Existence and Uniqueness Theorem. That theorem guarantees that when $f(t,x)$ is a k times continuously differentiable function of t,x then the flow $\phi(t,t_0,x)$ is k times continuously differentiable function of t,t_0,x. In addition, if $f(t,x,a)$ is C^k in its dependence of t,x and on parameters $a=(a_1,a_2,\ldots,a_k)$, then the flow is a C^k function of t,t_0,x,a.

Example 7.1 *An example is the parameter h in the logistic equation with periodic harvesting. A second example are the parameters $m>0$ and $k>0$ in the equation of a vibrating spring, $m\,x''+k\,x \;=\; 0$.*

Proposition 7.1 *For any equation* (1.1) *one has for all t,x, $\partial\phi/\partial x>0$.*

Proof. In equation (7.1) differentiate with respect to x. The computation is challenging at the level of notation for the chain rule. Denote $\partial_2 f$ for the partial derivative of f with respect to the second variable. To avoid confusion between the initial condition and the solution $x(t)$, denote the initial value by $\underline{x}$. The flow $\phi(t,\underline{x})$ then satisfies

$$\frac{\partial\phi(t,\underline{x})}{\partial t} \;=\; f(t,\phi(t,\underline{x}))\,, \qquad \phi(0,\underline{x}) \;=\; \underline{x}\,.$$

Differentiate with respect to $\underline{x}$ to find

$$\frac{\partial}{\partial\underline{x}}\frac{\partial\phi}{\partial t} \;=\; \partial_2 f\big(t\,,\,\phi(t,\underline{x})\big)\,\frac{\partial\phi}{\partial\underline{x}}\,, \qquad \frac{\partial\phi(0,\underline{x})}{\partial\underline{x}} \;=\; 1\,. \tag{7.1}$$

The equality of mixed partials implies that

$$z_1(t,\underline{x}) \;:=\; \frac{\partial\phi(t,\underline{x})}{\partial\underline{x}}$$

satisfies the linear ordinary differential equation

$$\frac{\partial}{\partial t}\,z_1 \;=\; \partial_2 f(t,\phi(t,\underline{x}))\,z_1\,, \qquad z_1(0,\underline{x}) \;=\; 1\,.$$

For each fixed $\underline{x}$ this is an ordinary differential equation in t even though the time derivative must be written with partials since z_1 depends on more than one variable.

[3] A summary of the method of perturbation theory is available on a separate posting in the Course Materials.

Such scalar linear ordinary differential equations are analysed using the **integrating factor** e^{-B}. Choosing B cleverly then multiplying the differential equation by e^B yields an exact derivative. Define

$$A(t,\underline{x}) \;:=\; \partial_2 f(t,\phi(t,\underline{x}))\,, \qquad B(t,\underline{x}) \;:=\; \int_0^t A(t,\underline{x})dt\,, \qquad \text{so}, \qquad \partial_t B = A\,.$$

Then

$$\partial_t\big(e^{-B}\,z_1\big) \;=\; e^{-B}\big(\partial_t z_1 - B_t z_1\big) \;=\; e^{-B}\big(\partial_t z_1 - A z_1\big) \;=\; 0\,.$$

Therefore $e^{-B}\,z_1$ is independent of t so,

$$e^{-B(t,\underline{x})}\,z_1(t,\underline{x}) \;=\; e^{B(0,\underline{x})}\,z_1(0,\underline{x}) \;=\; 1, \qquad \text{hence,} \qquad z_1 \;=\; e^B \;>\; 0\,.$$

This is the desired result. □

Proposition 7.2 *If $\partial_2^2 f < 0$ then for $t > 0$ $\phi(t,x)$ is strictly concave down in x.*

Proof. Differentiate (7.1) with respect to x to find[4]

$$\frac{\partial}{\partial \underline{x}}\frac{\partial}{\partial t} z_1 \;=\; \partial_2 f(t,\phi(t,\underline{x}))\frac{\partial}{\partial \underline{x}} z_1 \;+\; \partial_2^2 f(t,\phi)\,\phi_{\underline{x}}\,z_1\,.$$

Define

$$z_2(t,\underline{x}) \;:=\; \frac{\partial^2 \phi}{\partial \underline{x}^2} \;=\; \frac{\partial}{\partial \underline{x}} z_1\,,$$

to find the linear ordinary differential equation initial value problem

$$\frac{\partial}{\partial t} z_2 \;=\; A\,z_2 \;+\; \partial_2^2 f(t,\phi)\,\phi_{\underline{x}}\,z_1\,, \qquad z_2(0,x) \;=\; 0\,. \tag{7.2}$$

To show that ϕ is strictly concave down in x it suffices to prove that $z_2 < 0$. Since $z_1 := \phi_{\underline{x}}$, the source term in the ordinary differential equation (7.2) is $n(t) := \partial_2^2 f\, z_1^2$. The preceding proposition together with the strict concavity of f imply that $n < 0$ (n stands for negative). Thus,

$$z_2' = A\,z_2 \;+\; n(t)\,, \qquad z_2\big|_{t=0} \;=\; 0\,.$$

The integrating factor method yields,

$$\partial_t\big(e^{-B} z_2\big) \;=\; e^{-B}\big(\partial_t z_2 - A\,z_2\big) \;=\; e^{-B}\,n \;<\; 0.$$

[4]This is second order perturbation theory.

Thus $e^{-B}z_2$ is strictly decreasing and vanishes at $t = 0$. Therefore $e^{-B}z_2 < 0$ for $t > 0$. Since $e^{-B} > 0$ this completes the proof. □

Remarks. The same argument works with concave up instead of down. And also if strict concavity is replaced by concavity.

Example. For the logistic equation with periodic harvesting from §1.4, the Poincaré map for $t > 0$ is strictly increasing and strictly concave down.

8 Application to the logistic equation.

This section replaces a too brief discussion on the bottom of page 14 of [1]. Consider the logistic equation with harvesting

$$x' = ax(1-x) - h(1 + \sin 2\pi t), \qquad a > 0, \quad h > 0\,.$$

We fix a the ecological background, and discuss the effects of harvesting.

Denote by $\phi(t, x, h)$ the flow and $p(x, h) = \phi(1, x, h)$ the Poincaré map with harvesting rate h. Proposition 7.1 implies that $p_x > 0$. Compute $\partial_x^2 f = -2a < 0$. Proposition 7.2 implies that $p_{xx} < 0$. The next exercise shows that $\phi_h < 0$ establishing the intuitively clear assertion that increasing the harvesting rate, decreases the population for all $t > 0$. A more general comparison theorem of this type is given in the last section of this handout.

Exercise 8.1 *Denote by $\phi(t, x, h)$ the flow of the logistic equation with harvesting h,*

$$x' = ax(1-x) - h(1 + \sin 2\pi t)\,. \tag{8.1}$$

Use a perturbation argument like that above to show that for $h \geq 0$, x, and $t > 0$,

$$\frac{\partial \phi(t, x, h)}{\partial h} < 0\,. \tag{8.2}$$

The equation with $h = 0$ is autonomous and is well understood. The Poincaré map has two fixed points, $x = 0$ and $x = 1$. The function $p(x) - x$ is concave down with positive derivative at 0 and negative derivative at 1.

Since $p(x, h) - x$ is decreasing with h and concave down it follows that for h small and positive there is an unstable fixed point close to and to the right of 0 and a stable fixed point close to and to the left of 1.

Proposition 8.1 *For $h > a/4$ one has $p(x,h) - x < 0$ for all x.*

Proof. Compute

$$x(1) - x(0) = \int_0^1 x'(t)\ dt = \int_0^t a\,x(t)\,(1 - x(t)) - h(1 + \sin 2\pi t)\ dt\,.$$

The maximum in x of $x(1-x)$ is $1/4$. It follows that

$$x(1) - x(0)\ \le\ \int_0^t \frac{a}{4} - h(1 + \sin 2\pi t)\ dt\ =\ \frac{a}{4} - h\,.$$

When $h > a/4$ the right hand side is strictly negative. □

As h increases, the concave down curve $p(x,h) - x$ decreases. For $h = 0$ and nearby values the graph has an arc above the axis and there are exactly two equilibria. As h increases the graph decreases. By the time $h = a/4$ the curve lies entirely below axis and there are no equiliibria. There is an intermediate value $h_* \in]0, a/4]$ where $p(x,h_*) - x$ is nonpositive, with graph touching the axis at a unique equilibrium. The value h_* is a bifurcation. There are two equilibria for $h < h_*$ and none for $h > h_*$. For $h \in]0, h_*[$, the concavity shows that $p(x) - x$ is positive between its two roots and negative to the left of the smaller and to the right of the larger. It follows that larger periodic orbit is stable and the smaller is unstable. For the values of h the qualitative behavior is that suggested by figures 1.10 and 1.11. For $h > h_*$ the harvesting drives the population to zero in finite time.

Exercise 8.2 *Take $a = 1$. Use a computer to compute an approximate values of $p(x,h) - x$ for $h = a/8$ and regularly spaced initial data in $[0,1]$. If one finds $p(x,a/8) - x$ has roots then $h_* \in [a/8, a/4]$. Otherwise $h_* \in [0, a/8]$. One can continue this bisection process to find h_* as accurately as one wants.*

9 Structural stability.

If $p(x) - x$ has only a finite number of roots and at those roots $p' \neq 1$ then the curve $y = p(x) - x$ crosses the x-axis transversally at the roots.

A small perturbation of f on a bounded set of x yields a new Poincaré map $\widetilde{p} \approx p$ so $y = \widetilde{p}(x) - x$ is a small perturbation of $y = p(x) - x$.

Therefore $\widetilde{p}(x) - x$ will have roots near the old roots. The derivative test shows that they will have the same stability properties,. The long term dynamics will be essentially equivalent to that of the unperturbed problem. The original problem is **structurally stable.**

These short paragraphs should convince you of the power of the dynamical systems point of view.

Exercise 9.1 *With h_* the bifurcation value for the logistic equation with harvesting, show that for $0 < h < h_*$ the dynamics of the logistic equation with harvesting is structurally stable. Draw the same conclusion for $h > h_*$.*

10 Comparison principles.

The monontonicity principle is an example of a *comparison principle* [5] asserting that under suitable hypotheses one solution is smaller than another. The monotonicity result followed from uniqueness. The next results are subtle and far reaching generalizations.

Theorem 10.1 (Strict Comparison Theorem) *Suppose that $f(t,x)$ and $\tilde{f}(t,x)$ are continuously differentiable on $[a,b]\times\mathbb{R}$ and satisfy*

$$f(t,x) \ < \ \tilde{f}(t,x)\,, \qquad \text{for all} \qquad (t,x) \ \in \ [a,b]\times\mathbb{R}\,.$$

If $x(t)$ and $\tilde{x}(t)$ are solutions on $a \le t \le b$ of

$$x' = f(t,x), \qquad \tilde{x}' = \tilde{f}(t,x)\,, \qquad \text{with} \qquad x(a) \ < \ \tilde{x}(a),$$

then for all $t \in [a,b]$

$$x(t) \ < \ \tilde{x}(t)\,.$$

Proof. Must show that it is impossible that there is a $T \in [a,b]$ with $\tilde{x}(T) - x(T) \le 0$. Since $\tilde{x}(a) - x(a) > 0$, if such a T existed the intermediate value theorem would imply that there was a $\underline{t} \in]a,T]$ so that $\tilde{x}(\underline{t}) - x(\underline{t}) = 0$. In that case, $K := \{t \in [a,b] \,:\, \tilde{x}(t) - x(t) = 0\}$ would be a nonempty closed subset with $a \notin K$. It would therefore have a smallest element $\underline{t} > a$.

For $0 \le t \le \underline{t}$, $\tilde{x}(t) - x(t) > 0$ and $\tilde{x}(\underline{t}) - x(\underline{t}) = 0$ so

$$\tilde{x}'(\underline{t}) - x'(\underline{t}) \ = \ \lim_{n\to\infty} \frac{(\tilde{x}(\underline{t}) - x(\underline{t})) \ - \ (\tilde{x}(\underline{t} - 1/n) - x(\underline{t} - 1/n))}{1/n} \ \le \ 0 \tag{10.1}$$

[5] Student difficulties with problems 18/15 and 18/16 from Hirsh, Smale, and Devaney and problem 4 on the 2010 midterm have led to inclusion of this section.

since the numerator is ≤ 0.

On the other hand,

$$\tilde{x}'(\underline{t}) \;=\; \tilde{f}(\underline{t}, \tilde{x}(\underline{t})) \;>\; f(\underline{t}, \tilde{x}(\underline{t})) \;=\; f(\underline{t}, x(\underline{t})) \;=\; x'(\underline{t})\,. \tag{10.2}$$

The contradiction between (10.1) and (10.2) proves that no such T can exist. □

Exercise 10.1 *The proof uses only that $\tilde{f}(t, \tilde{x}(t)) > f(t, \tilde{x}(t))$ for all t. That give a result with hypotheses only along the curve $(t, x(t))$. What does $\tilde{f}(t, \tilde{x}(t)) > f(t, \tilde{x}(t))$ say about the direction field of f along the curve $(t, x(t))$? Sketch. Explain geometrically why this prevents a solution curve of the f equation from crossing from below to above $(t, \tilde{x}(t))$.*

Theorem 10.2 (Comparison Theorem) *Suppose that $f(t, x)$ and $\tilde{f}(t, x)$ are continuously differentiable on $[a, b] \times \mathbb{R}$ and satisfy*

$$f(t, x) \;\leq\; \tilde{f}(t, x)\,, \qquad \text{for all} \qquad (t, x) \;\in\; [a, b] \times \mathbb{R}\,.$$

If $x(t)$ and $\tilde{x}(t)$ are solutions on $a \leq t \leq b$ of

$$x' = f(t, x), \qquad \tilde{x}' = \tilde{f}(t, x)\,, \qquad \text{with} \qquad x(a) \;\leq\; \tilde{x}(a),$$

then for all $t \in [a, b]$,

$$x(t) \;\leq\; \tilde{x}(t)\,.$$

Proof. Define $x(t, \epsilon)$ to be the solution of

$$x' \;=\; f(t, x) - \epsilon\,, \qquad x(a, \epsilon) \;=\; x(a)\,.$$

For $\epsilon = 0$, $x(t, 0) = x(t)$.

The Continuous Dependence Theorem implies that there is an $\epsilon_0 > 0$ so that for $0 \leq \epsilon \leq \epsilon_0$, $x(t, \epsilon)$ exists for $a \leq t \leq b$ and depends continuously on ϵ.

The Strict Comparison Theorem implies that for $0 < \epsilon \leq \epsilon_0$ and $a \leq t \leq b$ $\tilde{x}(t) > x(t, \epsilon)$.

Passing to the limit $\epsilon \to 0$ yields

$$x(t) \;=\; \lim_{\epsilon \to 0} x(t, \epsilon) \;\leq\; \lim_{\epsilon \to 0} \tilde{x}(t) \;=\; \tilde{x}(t)\,.$$

□

Example 10.1 i. *Consider two solutions of harvesting model* (8.1) *with harvesting rates $h_1 > h_2$. Intuitively, larger harvesting rate should lead to smaller populations. The Strict Comparison Theorem shows that if $x_1(0) < x_2(0)$ then for all $t \geq 0$, $x_1 < x_2$.*

ii. *If one has only $h_1 \geq h_2$ and $x_1(0) \leq x_2(0)$ the Comparison Theorem implies that $x_1 \leq x_2$.*

Exercise 10.2 *In* **ii**, *show that if $h_1 > h_2$ and $x_1(0) \leq x_2(0)$ then for $t > 0$, $x_1 < x_2$.* **Hint.** *Only the case $x_1(0) = x_2(0)$ is needed. Use* (8.2). *Alternatively, prove the relation for t small positive and apply the Strict Comparison Theorem beyond that.*

Exercise 10.3 *In* **ii**, *show that if $h_1 \geq h_2$ and $x_1(a) < x_2(a)$ then for $t \geq 0$, $x_1 < x_2$.* **Hint.** Redo the proof of the Strict Comparison Theorem showing where it needs to be changed.

Remarks. 1. Generalizing these two exercises shows that if in the Comparison Theorem one has $f < \tilde{f}$ then for $a < t \leq b$ one has $x_1(t) < \tilde{x}(t)$. Similarly if $f \leq \tilde{f}$ and $x_1(a) < x_2(a)$ then for all $a \leq t \leq b$, $x_1(t) < x_2(t)$.

2. All the results and proofs extend to differential inequalities $x' \leq f(t, x)$ and $\widetilde{x}' \geq \widetilde{f}(t, x)$.

References

[1] M. Hirsch, S. Smale, and R.L. Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos* 3rd. ed., Elsevier, 2011.

[2] J. Hale and H. Kocak, *Dynamics and Bifurcations*, Springer-Verlag, 1991.