Dynamics in Dimension 1

1 Introduction.

This is a supplement to section 1.1, 1.2, 1.3 in [1]. There are good things to be learned from presenting this material more slowly. Part I of [2] is excellent but if we did that much detail we'd never get off the ground. These notes plus the sections of the text map out a middle road.

The reader should note how often one appeals to the Fundamental Existence and Uniqueness Theorem. Though I have chosen not to prove this result in 558, Its importance will be clear from the number of times that it is used.

2 Separation of variables. (suppl. to §1.1)

In first term calculus one learns the method of separation of variables to find formulas for solutions of scalar real equations of the form

$$x' \;=\; g(x)\, h(t)\,.$$

A classic example is the second most important differential equation of them all

$$\frac{dx}{dt} = a\,x, \qquad 0 \neq a \in \mathbb{R}\,.$$

The most important is $x' = 0$. The standard solution is to write

$$\frac{dx}{x} \;=\; a\,dt\,. \tag{2.1}$$

Integrate to find where $x > 0$,

$$\ln x \;=\; at + C$$

where the two constants of integration are lumped together. Exponentiate to find

$$x = Be^{at}, \qquad B = e^{C}\,.$$

It is easy to verify that the resulting functions solve the differential equation. The formula allows one to find exactly one solution of each initial value problem

$$x' = ax, \qquad x(t_0) = x_0\,.$$

with $x_0 > 0$. For the opposite sign one gets $-e^C\, e^{at}$. The Fundamental Existence Theorem asserts (under suitable hypotheses satisfied here) that each such initial value problem has exactly one solution. Since we have a solution we have them all.

So, at least in this case the magic method starting at (2.1) works. In fact the method is rigorous. Since $x(t) = 0$ is a solution, the Fundamental Uniqueness Theorem shows that if there is a time t_1 with $x(t_1) = 0$ then $x(t) = 0$ for all time. Therefore, if $x_0 \neq 0$ then $x(t) \neq 0$ for all time. If $x_0 > 0$ one has $x(t) > 0$ for all time, otherwise there would be a time with $x = 0$ and that has been ruled out. The chain rule implies that

$$a \;=\; \frac{1}{x}\,\frac{dx}{dt} \;=\; \frac{d}{dt}\ln x \;=\; \frac{d}{dt}(at)\,.$$

Thus $(\ln x - at)' = 0$ and the most important equation of all implies that $\ln x - at$ is constant.

Examine the logic. This argument shows that if $x(t)$ is a positive solution on an interval I, then $x = Be^{at}$ for some B. To show that $B\,e^{at}$ is a solution one either plugs it in, or more interestingly, observes that the derivation can be read in the opposite direction to show that if $\ln x - at$ is constant and x is differentiable then $a - x'/x = 0$.

A second important example of separation of variables is

$$x' = x^2, \quad x(0) = x_0, \qquad \text{with solution} \qquad x(t) \;=\; \frac{x_0}{1 - tx_0}\,.$$

If $x_0 > 0$ then the solution increases to $+\infty$ as t increases to $1/x_0$. The solution exists on the interval $-\infty < t < 1/x_0$. If $x_0 < 0$ the solution explodes at finite negative time.

This shows that the fundamental existence theorem can at best guarantee existence on a finite and possibly small interval containing the initial time t_0.

Exercise 2.1. *Suppose that $f > 0$ in $[x_1, x_2]$. Show that the solution of $x' = f(x)$ with $x(0) = x_1$ reaches x_2 at time*

$$T \;=\; \int_{x_1}^{x_2} \frac{1}{f(x)}\, dx\,.$$

Hint. Since $dx/dt > 0$ one can consider the inverse function $t = t(x)$. Compute its derivative. Equivalently, compute the time dt it takes to move from x to $x + dx$ and sum.

Exercise 2.2. *Derive the following characterization of finite time blowup to* $+\infty$. **Hint.** Use the preceding exercise.

Proposition. *The solution of the initial value problem* $x' = f(x)$, $x(0) = x_0$ *diverges to* $+\infty$ *in finite time if and only if* $f > 0$ *on* $[x_0, \infty[$ *and* $T_* := \int_{x_0}^{\infty} dx/f(x) < \infty$. *In this case the solution exists on* $[0, T_*[$ *and* $\lim_{t \nearrow T_*} x(t) = \infty$.

3 The fundamental theorem of the phase line.

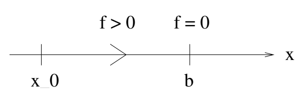
Supplement to §1.2. For the equation $x' = f(x)$ the following are equivalent,

• $\underline{x}$ satisfies $f(\underline{x}) = 0$. Equivalently, $\underline{x}$ is a **root** of the equation $f(x) = 0$.

• the graph of the function $y = f(x)$ meets the x axis at $\underline{x}$

• The function $x(t) = \underline{x}$ is an equilibrium solution.

The theorem concerns the solution of the equation $x' = f(x)$ with x_0 to the left of an equilibrium b and the phase line diagram pointing to the right on $[x_0, b[$.



Theorem 3.1 *Suppose that* $-\infty < x_0 < b < \infty$ *and* $f(x)$ *is a continuously differentiable function on* $[x_0, b]$, *with*

$$f(b) = 0, \qquad \text{and,} \qquad f(x) > 0 \quad \text{for} \quad x_0 \leq x < b\,.$$

Then the solution of the initial value problem

$$x' = f(x), \qquad x(0) = x_0$$

i. *Satisfies* $x_0 < x(t) < b$ *for all* $t > 0$ *and is strictly monotone increasing.*

ii. *Satisfies* $\lim_{t \to \infty} x(t) = b$.

Proof of Theorem. i. One must have $x(t) < b$ for all t. Otherwise the Intermediate Value Theorem for continuous functions would imply that there is a $T > 0$ with $x(T) = b$. The uniqueness theorem implies that if $x' = f(x)$ and $x(T) = b$ then $x(t) = b$ for all time violating the initial condition. Therefore $x(t) < b$ for all t. $x_0 < x < b$ on has $f(x) > 0$ so $x' = f(x) > 0$ implying strict monotonicity.

ii. Must show that for any $L < b$ there is a time $\underline{t}$ so that for $t > \underline{t}$ one has $x(t) > L$. Suppose that for $0 \le t \le t_1$ one has $x(t) \le L$. Define[1]

$$0 \ < \ m \ := \ \min_{x_0 \le x \le L} f(x)\,.$$

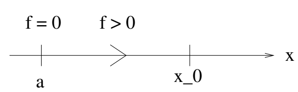
Then

$$L - x_0 > x(t_1) - x(0) = \int_0^{t_1} x'(s)\,ds = \int_0^{t_1} f(x(s))\,ds \ge \int_0^{t_1} m\,ds = t_1 m.$$

Thus $t_1 < (L - x_0)/m$. It is sufficient to take $\underline{t} := (L - x_0)/m$. □

Discussion. This beautiful little proof is hard to find elsewhere.

Exercise 4.1. *State an analogous fundamental results concerning the limit as $t \to -\infty$ and $f > 0$.* . The corresponding phase line diagram is



Hint. One approach is to repeat the proof of the Theorem. Alternatively, find the differential equation satisfied by $u(t) := x(-t)$ and apply the Theorem to $u(t)$. This is called **time reversal**.

Here is the analogous result for $t \to \infty$ and $f < 0$.

Corollary 3.2 *Suppose that $-\infty < b < x_0 < \infty$ and $f(x)$ is a continuously differentiable function on $[a, x_0]$, with*

$$f(a) = 0, \qquad \text{and,} \qquad f(x) < 0 \quad \text{for} \quad b < x \le x_0\,.$$

Then the solution of the initial value problem

$$x' = f(x), \qquad x(0) = x_0$$

i. *Satisfies $b < x(t) \le x_0$ for $t > 0$ and is strictly monotone decreasing.*

ii. *Satisfies* $\lim_{t\to\infty} x(t) = b$.

[1] We will be using many results from advanced calculus. Here we use the fact that a continuous function on a closed bounded interval attains a minimum value. For part **i** we used the intermediate value theorem for continuous functions.

Exercise 4.2. *Draw the phase line diagram for the Corollary and prove the corollary.* **Hint.** One approach is to repeat the argument of the Theorem. Alternatively, find a differential equation satisfied by $w(t) := -x(-t)$ and apply the Theorem to $w(t)$.

Exercise 4.3. *State, draw phase line, and prove the analogous result for* $t \to -\infty$ *and* $f < 0$.

Exercise 4.4. *Suppose that the graph of* f *has only a single hump between equilibria* a *and* b*, that is it rises strictly monotonically to a maximum and then falls strictly monotonically. Show that the solutions of* $x' = f(x)$ *whose values lie in* $]a, b[$ *consist of one arc convex up then an inflection then an arc that is convex down.* **Hint.** Differentiate the equation to find an expression for x'' in terms of x. Use the second derivative test for convexity.

Examples. Logistic and logistic with constant harvesting.

4 Asymptotic stability of equilibria. (suppl. to §1.2)

Definition 4.1 *A root* $\underline{x}$ *of* $f(x) = 0$ *is called a* **simple root** *if and only if* $f'(\underline{x}) \neq 0$

Since the derivative $f'(\underline{x})$ can be any real number it is unlikely that it vanishes. So, a typical root $\underline{x}$ is expected to be simple.

• If $f'(\underline{x}) < 0$ then f changes from positive to negative as x increases through $\underline{x}$.

• In that case, the fundamental theorems imply that solutions on either side of $\underline{x}$ converge to $\underline{x}$ as $t \to \infty$.

The point $\underline{x}$ "attracts" all nearby orbits and is called an **attracting equilibrium** or a **sink**. Such equilibria are called **asymptotically stable**.

In case $f'(\underline{x}) > 0$ the equilibrium attracts orbits as $t \to -\infty$. For increasing t the orbits are "repelled". The equilibrium is called a **repelling equilibrium** or **source** and is **unstable**.

By definition, equilibria are stable when all nearby initial conditions yield solutions that stay nearby. When orbits on at least one side are repelled the equilibrium is unstable.

5 Structural stability of equilibria.

Supplement to §1.2,1.3.

If $\underline{x}$ is a root of $f(x) = 0$ and $f'(\underline{x}) \neq 0$, then (and only then) the curves $y = 0$ and $y = f(x)$ cross at $(\underline{x}, 0)$ with unequal slopes. This is called a **transverse crossing**.

Proposition 5.1 *If the function $f(x)$ is slightly perturbed (that is replaced by $\widetilde{f}$ with $\widetilde{f} - f$ small together with its first derivatives) then the perturbed curve will intersect the x-axis at a point $\underline{\widetilde{x}} \approx \underline{x}$ where the sign of $\widetilde{f}'(\underline{\widetilde{x}})$ is equal to the sign of $f'(\underline{x})$.*

Proof. Treat the case $f'(\underline{x}) = c > 0$. In that case if $\alpha < \underline{x} < \beta$ are sufficiently close to $\underline{x}$ then

$$f(\alpha) < 0, \qquad f(\beta) > 0, \qquad \text{and,} \qquad \min_{x \in [\alpha,\beta]} f'(x) \ \geq \ \frac{c}{2}\,.$$

If $\widetilde{f}$ is so close to f that

$$\max_{[\alpha,\beta]} |f - \widetilde{f}| + \max_{[\alpha,\beta]} |f' - \widetilde{f}'| \ < \ \min\{|f(\alpha|, |f(\beta)|, c/2\},$$

then $\widetilde{f}$ is strictly increasing from a negative value at α to a positive value at β. The graph of $\widetilde{f}$ crosses the x axis at a unique equilibrium point in $[\alpha, \beta]$. $\Box$

An alternate method of proof appeals to the Implicit Function Theorem. We will treat the multidimensional case using that method.

The conclusion is that a *simple root* of $f = 0$ that is asymptotically stable is not destroyed by small perturbations of f. Such stability under small changes of the equation is called **structuraly stability**. In practical examples (think of mx'' where m is only known to a finite precision), conclusions invariant under such small perturbations are particularly important.

The example of the logistic equation with *constant harvesting* on pages 8-9 of the text with harvesting rate has $h = 1/4$ has an equilibrium that is NOT structurally stable. For h slightly smaller there are two nearby equilibria and for h slightly larger there are none. At $h = 1/4$ the root of $f(x) = 0$ is a double root of the corresponding quadratic equation. It is a **bifurcation point**. The case of time dependent periodic harvesting has a similar bifurcation as shown in the Dynamics in Dimension 1.5 handout.

Exercise 5.1. consider the equilibrium $x = 0$ of the logistic equation $x' = x(1-x)$. The perturbed equation

$$x' = x(1-x) + \epsilon \sin x \,, \qquad |\epsilon| << 1 \,,$$

has an equilibrium $x(\epsilon) \approx 0$. Find constants $c_j \in \mathbb{R}$ so that the equilibrium has the expansion

$$x(\epsilon) \;=\; 0 + c_1 \epsilon + c_2 \epsilon^2 \;+\; \text{higher order terms in } \epsilon \,.$$

Hint. The quickest way is to plug the expansion into the equation for the equilibrium and equate terms of the same order in ϵ. The more analytic was is to compute the derivative

$$\frac{d^j x}{d\epsilon^j}\bigg|_{\epsilon=0} \quad \text{for} \quad j = 1, 2$$

by implicit differentiation. **Discussion.** You will meet many computations of this sort in the Bifurcation Theory handout.

6 Global phase portrait.

Supplement to §1.3.

If $f(x) = 0$ has only a finite number of roots

$$-\infty \;<\; x_1 \;<\; x_2 \;<\; \cdots \;<\; x_N \;<\; \infty,$$

they decompose the real line into $N+1$ intervals, two of them unbounded. The function f has constant sign on each of the intervals. The motion is to the left (resp. right) where f is negative (resp. postive).

An equilibrium x_j is asymptotically stable in this case if and only if f is positive to the left and negative to the right of x_j. The origin for $x' = -x^3$ is an example of an asymptotically stable equililbrium that is not a simple root and is not structurally stable. Small perturbations $x^3 + ax^2 + bx + c$ with a, b, c small yield cubic polynomials with one, two, or three roots near 0.

Exercise 6.1. *Draw graphs of three different cubic polynomials* $f_1(x)$, $f_2(x)$, *and* $f_3(x)$, *each a small perturbation of* x^3, *so that* $x' = f_j(x)$ *has exactly* j *equilibria for* $j = 1, 2, 3$. **Hint.** The equilibria must be close to $x = 0$.

Exercise 6.2. *Continue the preceding exercise. Explain why the example with three equilibria is structurally stable while that with two equilibria is not.* **Discussion.** There are one equilibrium perturbations that are structurally stable, and, there are one equilibrium perturbations that are not structurally stable. You are invited to find such examples but are not required to.

Definition 6.1 *Two differential equations*

$$x' = f(x), \qquad x' = g(x)$$

are said to have **equivalent phase portraits** *when they have the same finite number of equilibria and the pattern of signs on successive intervals is the same.*

It is geometrically clear that if $f = 0$ has only simple roots and g is a perturbation that only changes f on a bounded interval, then for sufficiently small perturbations, the global phase portrait will not change. This property is called **(global) structural stability**.

Examples. Logistic and logistic with constant harvesting with the exceptional cases when the parabola is tangent to the x-axis.

Exercise 6.3 *For integer $N \geq 1$ define*

$$f_N(x) \ := \ x(x-1)(x-2) \ \cdots \ (x-N)\,.$$

For $N - 1 < x_0 < N$, let $x(t)$ be the solution of the initial value problem,

$$x' \ = \ f_N(x), \qquad x(0) = x_0\,.$$

Determine

$$\lim_{t\to\infty} x(t)\,.$$

Hint. The answer may be different for different values of N. **Discussion.** This is a good example where trying to determine the answer by finding a formula for $x(t)$ is a bad idea even though it is possible to find such a formula.

Exercise 6.4 *The equivalence of phase diagrams in the preceding example can miss some interesting behavior. It is concerned mostly with $|t|$ very large. That accounts for "most" times. But interesting things can occur for moderate times.*

Consider the function f from figure 1.12 shifted downward so that the peak at $x = 0$ is just barely below the x-axis. The resulting differential equation is called Equation (1). Equation (1) has phase portrait that is equivalent to the phase portrait of

$$x' = x^2 - 1\,. \tag{2}$$

Describe in words how the solutions of equation (1) with initial data to the left of and near the right hand equilibrium behave differently from the solutions of equation (2).

References

[1] M. Hirsch, S. Smale, and R.L. Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos* 3rd. ed., Elsevier, 2011.

[2] J. Hale and H. Kocak, *Dynamics and Bifurcations*, Springer-Verlag, 1991.