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Ellipse Axes. Aspect ratio, and, Direction of Rotation for Planar Centers

This handout concerns 2×2 constant coefficient real homogeneous linear systems $X' = AX$ in the case that A has a pair of complex conjugate eigenvalues $a \pm ib$, $b \neq 0$. The orbits are elliptical if $a = 0$ while in the general case, $e^{-at}X(t)$ is elliptical. The latter curves are the solutions of the equation

$$X' \;=\; (A - aI)X, \qquad a \;=\; \frac{\operatorname{tr} A}{2}\,.$$

For either elliptical or spiral orbits we associate this modified equation that has elliptical orbits. The new coefficient matrix $A - (\operatorname{tr} A/2)I$ has trace equal to zero and positive determinant. These two conditions characterize the matrices with a pair of non zero complex conjugate eigenvalues on the imaginary axis. Those are the matrices for which the phase portrait is a center. We show how to compute the axes of the ellipse, the eccentricity of the ellipse, and the direction of rotation, clockwise or counterclockwise.

§1. Direction of rotation.

To determine the direction of rotation it suffices to find the direction of the rotation on the positive x_1-axis. If the flow is upward (resp. downward) then the swirl is counterclockwise (resp. clockwise). For a matrix A that has no real eigenvalues, the direction of swirl is counterclockwise if and only if the second coordinate of

$$A\begin{pmatrix}1\\0\end{pmatrix} \;=\; \begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}$$

is positive, if and only if $a_{21} > 0$. The swirl is counterclockwise if and only if $a_{21} > 0$.

Freeing this computation from the choice $(1,0)$ introduces an interesting real quadratic form. For any real $X \neq 0$, the vectors X and AX cannot be parallel. Otherwise, X would be an eigenvector with real eigenvalue. Denote by $[X, AX]$ the 2×2 matrix whose first column is X and second is AX. Then the quadratic form

$$Q_1(X) \;:=\; \det[X, AX] \;=\; AX\cdot X^{\perp}, \qquad X^{\perp} \;:=\; (-x_2, x_1)\,.$$

is nonzero for all real $X \neq 0$. Therefore Q_1 is either always positive or always negative on $\mathbb{R}^2 \setminus 0$. The preceding criterion shows that the sign of $Q_1\big((1,0)\big)$ and therefore the sign of $Q_1(X)$ determines the direction of rotation.

This result can also be understood considering the angle θ in polar coordinates. On any curve $(x_1(t)\,,\,x_2(t))$ that does not touch the origin,

$$\frac{d\theta(x_1(t), x_2(t))}{dt} \;=\; \frac{\partial\theta}{\partial x_1}\,\frac{dx_1}{dt} \;+\; \frac{\partial\theta}{\partial x_2}\,\frac{dx_2}{dt} \;=\; \frac{x_1\,x_2' - x_2\,x_1'}{x_1^2 + x_2^2}\,, \tag{1.1}$$

the last equality involving the partials of θ.

Exercise 1.1. *Compute the partial deriviatives of* $\theta(x_1, x_2)$ *by differentiating the identity*

$$\|x\| \left(\cos\theta(x)\,,\, \sin\theta(x) \right) \;=\; (x_1\,,\, x_2)$$

with respect to x_1 *and* x_2*.*

Formula (1.1) applied to solutions of $X' = AX$ yeilds

$$\frac{d\theta}{dt} \;=\; \frac{x_1\, x_2' - x_2\, x_1'}{x_1^2 + x^2} \;=\; \frac{AX \cdot X^\perp}{|X|^2} \;=\; \frac{Q_1(X)}{|X|^2}\,.$$

Therefore Q_1 has a geometric interpretation,

$$Q_1(X) \;=\; |X|^2\, d\theta/dt\,. \tag{1}$$

The criterion for counterclockwise rotation is $d\theta/dt > 0$.

Algorithm I. $Q_1(X)$ *is a definite quadratic form and the direction of rotation is counterclockwise if and only if* $Q_1 > 0$ *on* $\mathbb{R}^2 \setminus 0$, *and clockwise if and only if* $Q_1 < 0$.

Replacing A by $A - \alpha I$ mulitplies the solutions of the differential equation by $e^{-\alpha t}$ and does not change $d\theta/dt$ since $X \cdot X^\perp = 0$. In particular,

$$Q_1(X) \;=\; AX \cdot X^\perp \;=\; (A - (\mathrm{tr}\, A/2)I)X \cdot X^\perp\,.$$

§2. Ellipse axis directions.

Define a second quadratic form associated to the matrix A with complex conjugate eigenvalues. First replace A by $A - (\mathrm{tr}A/2)I$ that has elliptical orbits and eigenvalues on the imaginary axis. The second quadratic form is defined by

$$Q_2(W) \;:=\; (A - (\mathrm{tr}A/2)I)W \cdot W\,.$$

The points where $(A - (\mathrm{tr}A/2)I)X \perp X$ *are the points where* $Q_2 = 0$*. For elliptical orbits, these are the directions of the principal axes.*

The plane is divided into four quadrants by these directions. In each quadrant $Q_2(W)$ has a fixed sign and changes sign exactly when v is along one of the principal axes.

Algorithm II. *For a center, the axes of the ellipse are the nonzero vectors W so that $Q_2(W) = 0$. When the ellipse is noncircular this gives a pair of perpendicular lines that are the direction of the principal axes of the ellipse.*

Examples. i. For

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}, \qquad Q_2(X) = AX \cdot X \qquad AX = (x_2, -4x_1), \qquad Q(X) = -3x_1x_2 .$$

Since $Q_2(W) = -3w_1w_2$, the equation of the axes is $x_1x_2 = 0$. The axes are the usual euclidean axes.

ii. For

$$A = \begin{pmatrix} 3 & 5 \\ -2 & -2 \end{pmatrix},$$

$\operatorname{tr} A = 3 - 2 = 1$, so

$$A - (\operatorname{tr} A/2)I = \begin{pmatrix} 2.5 & 5 \\ -2 & -2.5 \end{pmatrix}$$

is the trace free matrix whose motion is a center. The equation determining the axes is

$$0 = Q_2(W) = 2.5\, w_1^2 + 3w_1w_2 - 2.5\, w_2^2 .$$

Any nonzero solution must have $w_1 \neq 0$. Dividing by w_1 shows that solutions are all multiples of $W = (1, y)$ where,

$$2.5 + 3y - 2.5\, y^2 = 0, \qquad \text{equivalently,} \qquad 5y^2 - 6y - 5 = 0 . \tag{2}$$

The roots are

$$\frac{6 \pm \sqrt{6^2 - 4(-5)(5)}}{10} = \frac{6 \pm \sqrt{136}}{10} = \frac{3 \pm \sqrt{34}}{5} .$$

The two roots yield the directions $(1, y_1)$ and $(1, y_2)$ of the two axes of the ellipse. The orthogonality of the directions is equivalent to the fact that the product of the roots is equal to -1. This follows from (2) since in the quadratic equation for y, the constant term and the coefficient of y^2 differ by a factor -1.

iii. In the degenerate case where there are more than two directions for which $Q_2 = 0$, one has a quadratic equation with more than two roots so the quadratic form vanishes identically and Q_2 is identically equal to zero. In this case the orbits are circular.

§3. Major and minor axes.

The differential equation with coefficient $\widetilde{A} := A - (\operatorname{tr} A/2)I$ has elliptical orbits. Compute orthogonal unit vectors W_j along the axes of the associated ellipse using Algorithm II. Denote by Y the coordinates with respect to the basis W_j,

$$X = y_1W_1 + y_2W_2 , \qquad \text{with,} \qquad y_1 = X \cdot W_2, \quad y_2 = X \cdot W_2 . \tag{3.1}$$

Introduce the matrix T whose first column is W_1 and second column is W_2 so

$$T := \begin{pmatrix} W_{1,1} & W_{2,1} \\ W_{1,2} & W_{2,2} \end{pmatrix}.$$

Equation (3.1) asserts that

$$X = TY, \qquad \text{equivalently} \qquad Y = T^{-1}X. \tag{3.2}$$

From the definition of T one has

$$T(1,0) = W_1, \quad T(0,1) = W_2, \qquad \text{so,} \qquad T^{-1}W_1 = (1,0), \quad T^{-1}W_2 = (0,1). \tag{3.3}$$

Exercise 3.1 i. *If the columns of a 2×2 matrix M form an orthonomal basis of $\mathbb{R}^2$ show that $M^\dagger M = I$ where $\dagger$ denotes transpose.* **ii.** *Use this to show that $(T^\dagger T = I$. Derive from this that $T^{-1}(T^{-1})^\dagger = (T^{-1})^\dagger T^{-1} = I$.*

$X(t)$ satisfies $X' = \widetilde{A}X$ if and only if $Y = TX$ satisfies $Y' = T^{-1}\widetilde{A}TY$. Next compute $T^{-1}\widetilde{A}T$ to show that

$$T^{-1}\widetilde{A}T = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}, \qquad \text{with} \quad \alpha := \widetilde{A}W_2 \cdot W_1, \qquad \beta := \widetilde{A}W_1 \cdot W_2. \tag{3.4}$$

The first column of $T^{-1}\widetilde{A}T$ is equal to

$$T^{-1}\widetilde{A}T(1,0) = T^{-1}\widetilde{A}W_1.$$

Expand $\widetilde{A}W_1$ with respect to the orthonormal basis of W's to find

$$\widetilde{A}W_1 = (\widetilde{A}W_1 \cdot W_1)\, W_1 + (\widetilde{A}W_1 \cdot W_2)\, W_2 = \beta W_2.$$

Since $(\widetilde{A}W_1 \cdot W_1) = 0$ one has

$$T^{-1}\widetilde{A}W_1 = \beta T^{-1}W_2 = (0,\beta)$$

verifying the first column of (3.4).

Exercise 3.2. *Verify the second column of the identity.*

The formulas show that α and β are real. The determinant of $\widetilde{A}$ is positive and equal to the determinant of $T^{-1}\widetilde{A}T$ so $\alpha\beta < 0$ proving *that α and β have opposite sign.*

The differential equation in Y coordinates is

$$y_1' = \alpha y_2, \qquad y_2' = \beta y_1.$$

Multiply the first equation by $\beta\, y_1$, the the second by αy_2, and subtract to find

$$0 \;=\; \beta y_1 y_1' \;-\; \alpha y_2 y_2' \;=\; \frac{1}{2}\,\frac{d(\beta y_1^2 - \alpha y_2^2)}{dt}.$$

The orbits have equation $\beta y_1^2 - \alpha y_2^2 = \text{constant}$. The orbits are ellipses. Length of the axis in the y_1 direction divided by the axis in the y_2 direction is equal to $|\alpha/\beta|^{-1/2}$.

Algorithm III. *Suppose that W_j are orthogonal unit vectors along the axis directions found in Algorithm II and α and β are computed from formula (3.4). Then the elliptical orbits are similar to the ellipse with axis along W_1 of length $|\beta|^{-1/2}$ and axis along W_2 of length $|\alpha|^{-1/2}$.*

Examples. i. Choose $W_1 = (1,0)$, $W_2 = (0,1)$ unit vectors along the axes computed in §2. Then

$$\beta = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -4$$

$$\alpha = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\,.$$

The ellipses have axes along the x_1 and x_2 axis. The major axis is along x_2 and is longer by a factor 2 than the minor axis.

ii. Choose,

$$U_1 = \left(1\,,\; \frac{3+\sqrt{34}}{5}\right), \qquad W_1 = \frac{U_1}{\|U_1\|}, \qquad U_2 = \left(1\,,\; \frac{3-\sqrt{34}}{5}\right), \qquad W_2 = \frac{U_2}{\|U_2\|}\,.$$

Then

$$\beta = \frac{1}{\|U_1\|\|U_2\|} \begin{pmatrix} 2.5 & 5 \\ -2 & -2.5 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{3+\sqrt{34}}{5} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \frac{3-\sqrt{34}}{5} \end{pmatrix},$$

$$\alpha = \frac{1}{\|U_1\|\|U_2\|} \begin{pmatrix} 2.5 & 5 \\ -2 & -2.5 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{3-\sqrt{34}}{5} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \frac{3+\sqrt{34}}{5} \end{pmatrix},$$

The ellipses are similar to the ellipse with axes along W_1 and W_2 of lengths $|\beta|^{-1/2}$ and $|\alpha|^{-1/2}$ respectively.

Exercise 3.3. *If the axes are not along the x-axes, then the equation for $x = x_1/x_2$ in algorithm II has the form $x^2 + ax - 1 = 0$. Show that it is impossible to find an example where the roots are integers.* **Hint.** The sum of the roots is equal to $-a$ and the discriminant must be a perfect square.

This may explain why you don't find algorithms II, III in textbooks.