Gradient Systems

Summary. These notes complement the excellent section 9.3 of Hirsch, Smale, and Devaney.

1 Basic concepts

1.1 Definition

Definition 1.1 *Gradient systems are differential equations that have the form*

$$X' \;=\; -\mathrm{grad}\, V(X)\,,$$

with V *a real valued function.*

To guarantee that the right hand side is a continuously differentiable function of X one requires that V is twice continuously differentiable.

1.2 V decreases and steepest descent

The reason that gradient systems are grouped with the study of Lyapunov functions is that for gradient systems there is a natural candidate for such a function, Indeed, on solutions one has

$$V(X(t))' \;=\; \nabla V(X)\cdot X' \;=\; \nabla V(X)\cdot\big(-\nabla V(X)\big) \;=\; -\,\|\mathrm{grad}\, V(X(t))\|^2\,.$$

Except at equilibria, V *is strictly decreasing on orbits.*

The direction $\mathrm{grad}\, V(X)$ is the direction of most rapid increase of V. It is orthogonal to the level sets of V, The direction $-\mathrm{grad}\, V(X)$ is the direction of most rapid decrease of V. The orbits follow the path of steepest descent of V. If $V(X)$ represents altitude, then a skier who follows the fall line at all points follows these paths of steepest descent.

The numerical algorithm that seeks minima of $V(X)$ by descending toward the bottom of the graph of V on such curves is called the *method of steepest descent.*

1.3 How to recognize a gradient system

A differential equation

$$X' \;=\; F(X) \;=\; \Big(F_1(X), F_2(X), \dots, F_N(X)\Big)$$

is a gradient system if and only if there is a scalar valued function $V(X)$ so that

$$-\Big(F_1(X), F_2(X), \dots, F_N(X)\Big) \;=\; \left(\frac{\partial V(X)}{\partial x_1}\,,\ \frac{\partial V(X)}{\partial x_2}\,,\ \dots\,,\ \frac{\partial V(X)}{\partial x_N}\right).$$

In dimension $d = 1$

$$x' \;=\; f(x)$$

one can *always* choose an antiderivative V of $-f$ so that

$$\frac{dV(x)}{dx} \;=\; -f(x)\,.$$

The equation is *always* a gradient system

$$x' \;=\; -\frac{dV(x)}{dx}\,.$$

Exercise 1.1 *In dimension $d = 1$ the equations take the confusing form $x' = -V'(x)$ with V a function of the scalar variable. Explain why one cannot integrate to find that $x = V(x) + C$.*

In dimention $d = 2$ a system

$$x' \;=\; f(x,y)\,, \qquad y' \;=\; g(x,y)$$

is a gradient sytem if and only if there is a $V(x, y)$ so that

$$\frac{\partial V(x,y)}{\partial x} \;=\; -\,f(x,y)\,, \qquad \frac{\partial V(x,y)}{\partial y} \;=\; -\,g(x,y)$$

A necessary and sufficent condition for solvability on a ball $\{|X - \underline{X}| < R\}$ ($R = \infty$ allowed) is the equality of mixed partials,

$$\frac{\partial f}{\partial y} \;=\; \frac{\partial g}{\partial x}\,.$$

In the general case the necessary and sufficient condition on balls is again equality of mixed partials expressed as

$$\text{for all}\;\; 1 \le i < j \le N\,, \qquad \frac{\partial F_i(X)}{\partial x_i} \;=\; \frac{\partial F_j(X)}{\partial x_i}\,.$$

2 Equilibria and their stability

Equilibria are the **critical points** of V, also called **stationary points**, where the gradient of V is equal to zero.

Proposition 2.1 (HSD p.204) *If Z_0 is an ω or α limit point of a solution $X(t)$ of a gradient system then Z_0 is an equilibrium.*

Proof. Follows the proof of the LaSalle Invariance Principal, If $X(t_n) \to Z_0$ with $t_n \nearrow \infty$ consider the sequence of solutions $X_n(t) := X(t_n + t)$ that converge to the solution $Z(t)$ with initial value Z_0. Then $V(Z(t))$ is constant. This implies that Z_0 is an equilibrium so $Z(t) = Z_0$ for all t. ∎

Exercise 2.1 *Show that $V(Z(t)$ is constant. Show that V is constant on the ω-limit set of any orbit. Show that V is constant on the α-limit set of any orbit.* **Discussion.** It is challenging to find an example with a limit set larger than a single point.

2.1 Analysis of equilibria by linearization

If X^* is an equilibrium, the linearization is

$$Y' = -D^2V(X^*)\,Y\,,$$

where D^2V denotes the $N \times N$ symmetric matrix of second derivative,

$$D^2V(X^*) := \frac{\partial^2 V}{\partial x_i \partial x_j}(X^*)\,.$$

This matrix is called the *Hessian.* Thanks to the symmetry its eigenvalues are real. The next result follows from the standard linearization criteria paying special attention to the minus sign in the linearized equation.

Theorem 2.1 **i.** *If the eigenvalues of the Hessian are all strictly positive then, X^* is a sink. In particular, it is asymptotically stable.*

ii. *If the Hessian has a negative real eigenvalue then the equilibrium is unstable.*

iii. *If the eigenvalues of the Hessian are nonzero, then the dimension of the unstable manifold is equal to the number of negative eigenvalues counting mulitplicity. The dimension of the unstable manifold is the number of negative eigenvalues counting mulitplicity. The tangent spaces of these manifolds are the spans of the corresponding eigenspaces so are orthogonal at the equilibrium point.*

Remark 2.1 *Strictly positive eigenvalues as in* **i** *is the classic second derivative sufficient condition for a strict local minimum of V.*

Example 2.2 *The example (with 27 continuous derivatives)*

$$V(X) := |X|^{28}\Big(1 + \frac{\sin^2(1/|X|)}{10}\Big)\,,$$

has a strict local minimum but there are circles $|X| = r_k$ with $r_k \to 0$ consisting of equilibria, so X^* is NOT asymptotically stable. There are even circles of stable equilibria. Indeed, near the origin the first factor changes little on the scale at which the second varies so there are circles of stable equilbria near the minima of the second factor that occur at radii so that $1/r$ is an integer multiple of π.

2.2 Analysis of equilibria by the Lyapunov/Lasalle method

The function

$$L(X) := V(X) - V(X^*)\,,$$

decreases on orbits and vanishes at X^*.

Theorem 2.3 *L is a Lyapunov function when (and only when) X^* is a strict local minimum of V. It is a strict Lyapunov function when in addition X^* is an isolated critical point point of V. Lyapunov's Theorem implies that in the first case, X^* is stable, and in the second it is asymptotically stable.*

Remark 2.2 **i.** *Example 2.2 shows that the isolation hypothesis in the second assertion is needed.* **ii.** *X^* can be strict local minimum in cases where D^2V is not positive definite. For example $V(X) := \|X\|^4$.*

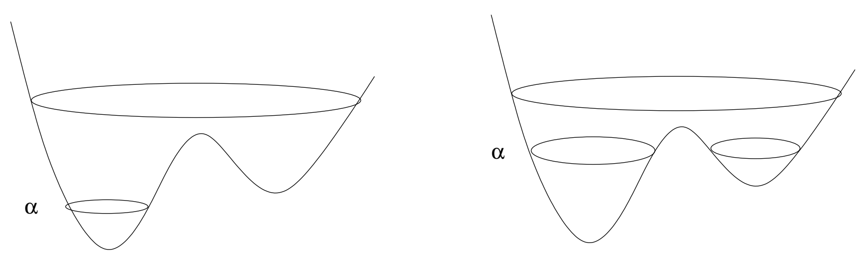
It is commonly thought that if $\nabla V(\underline{X}) = 0$ and V does **not** have a local minimum then the equilibrium is unstable. The idea is that orbits escapes from $\underline{X}$ by following paths along which V decreases. Unfortunately this appealing argument is incorrect as the next example shows.

Exercise 2.2 *Show that the one dimensional gradient system with $V(x) := x^{28}\sin(1/x)$ has $x = 0$ as a stable equilibrium and $x = 0$ is* **not** *a local minimum of $V(x)$.* **Hint.** The potential function $V(x)$ has lots of little positive barriers near $x = 0$.

Lasalle's Principle yields an accurate lower bound for the basin of attraction. Since L decreases on orbits it follows that for any α the set $\{X : V(X) \le \alpha\}$ is positively invariant. The strategy is to gradually increase α and find the largest sets to which Lasalle's principle applies.

Example 2.4 *In the left hand figure α is a little larger than the minimum of V and the region $\{V(X) \le \alpha\}$ is indicated by the region filled in the the bottom of the left hand bowl. The set is a connected neighborhood of the left hand minimum.*

In the second figure the region $\{V(X) \le \alpha\}$ is sketched for a larger value of α and the set of X includes two connected components one about each of the two local minima.



Each of the connected components is closed, bounded, positively invariant. For each of the two components, the only orbit in the component on which L is constant is the equlibrium in the set. This verifies Hypothesis **ii** *of Lassalle's Theorem.*

The hypothesis ceases to be satisfied when α is increased to the height of the saddle point in the graph of V. At that critical value of α, the two components merge and $\{X : V(X) \le \alpha\}$ becomes connected. That connected set contains three equilibria, both of the minima and the saddle point. In particular, the saddle point is an orbit on which L is constant. And, that orbit is not in the basis of attraction of either of the two minima.

The analysis of the example for α lower than the saddle yields the following general result.

Theorem 2.5 *If X^* is a strict local minimum of V with $V(X^*) < \alpha$ and if the connected component of X^* in $\{X : V(X) \le \alpha\}$ is compact and contains no equilibria other than X^*, then that component is contained in the basin of attraction of X^*.*

This proves the intuitive result that orbits starting in the bowl defined by the local minimum X^* descend the bowl to X^* so long as $V < \alpha$ at the initial point.

Exercise 2.3 The complete phase portrait of one such example is on page 205 of Hirsch, Smale, and Devaney. *For the asymmetric potential in Example 2.4 sketch the form that the stable and unstable manifold take.* **Hint.** You cannot find them exactly.