Theory of Linear Systems

This proves the two fundamental results concerning the theory of homogeneous linear systems

$$X' \;=\; A(t)\,X \;+\; f(t)\,.$$

The coefficient $A(t)$ is given as a continuous $N \times N$ matrix valued function on this interval and f is continuous both for $0 \le t < \infty$.

1 No blow up theorem

Extending A to be independent of t for $-\infty < t \le 0$ and for $T \le t < \infty$ we may suppose that A is defined and continuous for all t. The equation then has the form $X' = F(t, X)$ with F continuous in t and continuously differentiable in X on all of $\mathbb{R} \times \mathbb{R}^N$. The fundamental existence and uniqueness theorem implies that for any $X_0 \in \mathbb{R}^N$, the there is a unique maximal solution $X(t)$ defined for $0 \le t \le T_*$ satisfying the initial condition $X(0) = X_0$. In the nonlinear case one can have $T_* < \infty$. For linear equations this cannot occur unless the coefficients are singular.

Theorem 1.1 $T_* = \infty$. *In particular the solution exists throughout the interval of interest* $0 \le t \le T$.

Proof. The fundamental existence theorem shows that if $T_* < \infty$ then for any closed bounded set $\Gamma \in \mathbb{R}^N$ there is a $\underline{t} < T_*$ so that the solution takes values outside Γ for $t \in [\underline{t}, T_*[$. We show that this is not possible by finding an R so that for $0 \le t < T_*$ one has $\|X(t)\| \le R$. With $\Gamma := \{|X| \le R\}$ the blow up criterion is violated.

The strategy is to bound the growth of the square of the norm. Toward that end compute the derivative,

$$\big\langle X(t)\,,\, X(t)\big\rangle' \;=\; \big\langle X'(t)\,,\, X(t)\big\rangle \;+\; \big\langle X(t)\,,\, X'(t)\big\rangle\,.$$

Apply the Cauchy-Schwartz inequality twice to find

$$\big\langle X(t)\,,\, X(t)\big\rangle' \;\le\; 2\,\|X\|\,\|X'\|\,.$$

Estimate

$$\|X'\| \;=\; \|AX \;+\; f\| \;\le\; \|AX\| \;+\; \|f\| \;\le\; \|A(t)\|\|X\| \;+\; \|f(t)\|\,.$$

Define

$$a \;:=\; \max_{[0,T_*]} \|A(t)\|\,, \qquad b \;:=\; \max_{[0,T_*]} \|f(t)\|$$

Combining yields

$$\big\langle X(t)\,,\, X(t)\big\rangle' \;\le\; 2a\|X(t)\|^2 + 2b\|X\|\,.$$

Estimate

$$2b\|X\| \;\le\; b^2 + \|X\|^2$$

so $\varphi(t) \;:=\; \big\langle X(t)\,,\, X(t)\big\rangle$ satisfies

$$\varphi' \;\le\; (2a+1)\varphi + b^2\,.$$

Employing the method of integrating factors,

$$\Big(e^{-(2a+1)t}\varphi\Big)' \;=\; e^{-(2a+1)t}\varphi' - (2a+1)e^{-(2a+1)t}\varphi \;=\; e^{-(2a+1)t}\Big(\varphi' - (2a+1)\varphi\Big) \;\le\; b^2 e^{-(2a+1)t}.$$

Integrate from $t=0$ to t to find

$$e^{-(2a+1)t}\varphi(t) \;-\; \varphi(0) \;\le\; \int_0^t b^2 e^{-(2a+1)t}\,dt\,.$$

Therefore for $0 \le t < T_*$ multiplying by $e^{(2a+1)t}$ shows that

$$\|X(t)\|^2 \;=\; \varphi(t) \;\le\; e^{(2a+1)T_*}\Big(\varphi(0) \;+\; \int_0^{T_*} b^2 e^{-(2a+1)t}\,dt\Big) \;:=\; R^2\,.$$

completing the proof. ■

Alternate Proof. The fundamental existence theorem shows that if $T_* < \infty$ then for any closed bounded set $\Gamma \in \mathbb{R}^N$ there is a $\underline{t} < T_*$ so that the solution takes values outside Γ for $t \in [\underline{t}, T_*[$. We show that this is not possible by finding an R so that for $0 \le t < T_*$ one has $\|X(t)\| \le R$. With $\Gamma := \{|X| \le R\}$ the blow up criterion is violated.

The fundamental theorem of calculus implies that for $0 \le t < T_*$

$$X(t) - X(0) = \int_0^t X'(s)\,ds = \int_0^t A(s)\,X(s)\,ds\,.$$

The triangle inequality then implies that

$$\|X(t)\| \le \|X_0\| + \|\int_0^t A(s)\,X(s)\,ds\| \le \|X_0\| + \int_0^t \|A(s)\,X(s)\|\,ds\,.$$

Let

$$K := \max_{0\le t\le T} \|A(t)\| < \infty, \qquad \text{and} \qquad C := \|X_0\|\,.$$

K is finite because of the continuity of A and the fact that $[0,T]$ is closed and bounded. Then $\|A(s)X(s)\| \le K\|X(s)\|$ so $u(t) := \|X(t)\|$ satisfies for $0 \le t < T_*$

$$u(t) \le C + K\int_0^t u(s)\,ds\,.$$

Since u is continuous and nonnegative Gronwall's lemma implies that for $0 \le t < T_*$ one has

$$u(t) \le C\,e^{Kt}\,.$$

Setting $R := C\,e^{KT_*}$ proves the desired conclusion. ■

2 Theorem on general solutions

The superposition principal asserts that if $\Phi_j(t)$ for $1 \le j \le k$ are k solutions and c_j are k scalars then

$$c_1\Phi_1(t) + c_2\Phi_2(t) + \cdots + c_k\Phi_k(t)$$

is a k parameter family of solutions. The general solution is determined by its initial condition so is an N parameter family of solutions suggesting that for $k = N$ one might get the general solution. The next result tells under what conditions, $k = N$ solutions yield the general solution of the homogeneous linear system.

Theorem 2.1 *If $\Phi_j(t)$ is a solution for $1 \le j \le N$ then the following are equivalent.*

1. *For some $t_0 \in [0,T]$ the N vectors $\Phi_1(t_0), \Phi_2(t_0), \ldots, \Phi_N(t_0)$ are linearly independent in $\mathbb{R}^N$.*

2. *The general solution is $\sum c_j \Phi_j(t)$ with scalar c_j.*

3. *For all $\underline{t} \in [0,T]$, the vectors $\Phi_1(\underline{t}), \Phi_2(\underline{t}), \ldots, \Phi_N(\underline{t})$ are linearly independent in $\mathbb{R}^N$.*

Remark 2.1 *When the conditions hold, every solution is of the form $\sum c_j \Phi_j$ for a unique choice of c_j. The uniqueness of the c_j follows upon representing $X(t_0) = \sum c_j \Phi_j(t_0)$. Thanks to* **1**, *the vector $X(t_0)$ has only one such representation.*

Proof. Prove that $3 \Rightarrow 1 \Rightarrow 2 \Rightarrow 3$. That $3 \Rightarrow 1$ is trivial.

$1 \Rightarrow 2$. The superposition principal shows that each $\sum c_j \Phi_j$ is a solution.
Conversely if X is a solution consider the vector $X(t_0)$. Since the $\Phi_j(t_0)$ are N linearly independent vectors in $\mathbb{R}^N$ they span so there are scalars c_j so that

$$X(t_0) = c_1 \varphi_1(t_0) + c_2 \Phi_2(t_0) + \cdots + c_N \Phi_n(t_0)\,.$$

Define

$$Z(t) := c_1 \varphi_1(t) + c_2 \Phi_2(t) + \cdots + c_N \Phi_n(t)\,.$$

Then Z is a solution and $Z(t_0) = X(t_0)$ because of the choice of the c_j. Uniqueness implies that $Z = X$ so X is one the solutions $\sum c_j \Phi_j$. This completes the proof of 2.

$2 \Rightarrow 3$. Fix $\underline{t} \in [0,T]$. For any $W \in \mathbb{R}^N$ let $X(t)$ be the unique solution with $X(\underline{t}) = W$. Property 2 implies that there are constants c_j so that $X = \sum_j c_j \Phi_j$.
Evaluating at $t = \underline{t}$ implies that $\sum_j c_j \Phi(\underline{t}) = X(\underline{t}) = W$. Since W is arbitrary this shows that the set of vectors $\Phi_j(\underline{t})$ spans $\mathbb{R}^N$. A set of N vectors in $\mathbb{R}^N$ that spans is independent. This proves 3. ■