Linearization at an Equilibrium II. Perturbation Theory Approach

The first derivation of linearization is by approximating

$$F(X) = (f_1(X), f_2(X), \ldots, f_N(X))$$

for X approximately equal to an equilibrium $\underline{X}$ using Taylor's Theorem

$$F(X+\delta X) \;\approx\; F'(\underline{X})\delta X, \qquad F'(\underline{X}) \;:=\; \begin{bmatrix} \frac{\partial f_1(\underline{X})}{\partial X_1} & \cdots & \frac{\partial f_1(\underline{X})}{\partial X_n} \\ \frac{\partial f_2(\underline{X})}{\partial X_1} & \cdots & \frac{\partial f_2(\underline{X})}{\partial X_n} \\ & \cdots & \\ & \cdots & \\ & \cdots & \\ \frac{\partial f_n(\underline{X})}{\partial X_1} & \cdots & \frac{\partial f_n(\underline{X})}{\partial X_n} \end{bmatrix}.$$

This approximation drops terms higher order in δX.

The same result is obtained here using Perturbation Theory. The theorem on differentiable dependence then gives a rigorous justification of the linearization method.

Introduce a small parameter ϵ writing the initial value as $\underline{X} + \epsilon Z$,

$$X'(t,\epsilon) = F(X(t,\epsilon)), \qquad X(0,\epsilon) = \underline{X} + \epsilon Z. \tag{1}$$

The solution $X(t,\epsilon)$ is a continuously differentiable function of t, ϵ provided that F is continuously differentiable. Perturbation theory finds the Taylor approximation

$$X(t,\epsilon) \;\approx\; X(t,0) \;+\; \epsilon \frac{\partial X(t,0)}{\partial \epsilon}\,. \tag{2}$$

Define

$$X_0(t) \;:=\; X(t,0)\,, \quad X_1(t) \;:=\; \frac{\partial X(t,0)}{\partial \epsilon}, \qquad \text{so} \quad X(t,\epsilon) \approx X_0(t) + \epsilon X_1(t) \tag{3}$$

is the first order Taylor approximation. When F is twice continuously differentiable, $X(t,\epsilon)$ is then twice differentiable and Taylor's theorem with remainder shows that the error in the approximation (2) is not larger than $C\epsilon^2$ where C is a bound on the second derivative. In particular it is finite for bounded time intervals $0 \le t \le T$.

Setting $\epsilon = 0$ yields the equation for the unperturbed solution $X_0(t) := X(t,0)$,

$$X_0'(t) = F(X_0(t)), \qquad X_0(0) = \underline{X}. \tag{4}$$

Therefore, the unperturbed solution is the equilibrium,

$$X_0(t) \;=\; \underline{X}\,. \tag{5}$$

Differentiate (1) with respect to ϵ to find for all t, ϵ,

$$\frac{\partial}{\partial t}\frac{\partial X}{\partial \epsilon} \;=\; F'(X)\,\frac{\partial X}{\partial \epsilon}. \tag{6}$$

Differentiate the initial condition, (2), with respect to ϵ to find,

$$\frac{\partial X(0,\epsilon)}{\partial \epsilon} \;=\; Z\,. \tag{7}$$

Set $\epsilon = 0$ in (4),(5) using (3), to find the linearized initial value problem determining $X_1(t)$,

$$X_1'(t) \;=\; F'(\underline{X})\,X_1(t), \qquad X_1(0) = Z\,. \tag{8}$$

The approximation $\delta X :\approx \epsilon X_1(t)$ satisfies

$$\delta X' \;=\; F'(\underline{X}))\,\delta X\,. \tag{9}$$

This is the same result obtained by ignoring terms in δX of order higher than 1 in the science text style derivation given earlier. That derivation also suggests that the error is $\sim |\delta X|^2 \sim \epsilon^2$ that is rigorously established by the Perturbation Theory approach.