The Lyapunov and Lasalle Theorems

Hypothesis Suppose that X^* is an equilibrium of the system $X' = F(x)$ with F continuously differentiable.

Definition 1 *A continuously differentiable real valued function L from on an open neighborhood $\mathcal{O} \ni X^*$ is a* **lyapunov function** *when it has the following two properties.*

i. $L > 0$ *on* $\mathcal{O} \setminus X^*$ *and* $L(X^*) = 0$.

ii $\nabla_X L(X) \cdot F(X) \leq 0$ *on* $\mathcal{O}$.

It is a **strict lyapunov function** *when in addition*

iii. $\nabla_X L(X) \cdot F(X) < 0$ *on* $\mathcal{O} \setminus X^*$.

Property **ii** asserts that L is nondecreasing on orbits in $\mathcal{O}$. Property **iii** asserts that the time derivative of L on orbits in $\mathcal{O} \setminus X^*$ is strictly negative.

The first two theorems are due to Lyapunov. The last two are called LaSalle's Invariance Principal.

Theorem 1 *If there exists a lyapunov function, then the equilibrium X^* is stable*

Theorem 2 *If there exists a strict lyapunov function, then the equilibrium X^* is asymptotically stable*

Theorem 3 *Suppose that L is a Lyapunov functional on $\mathcal{O}$ and $X(t)$ is an orbit lying in a closed bounded set $K \subset \mathcal{O}$. If Z_0 is an ω-limit point of $X(t)$ and $Z(t)$ is the orbit with $Z(0) = Z_0$, then $Z(t)$ lies in K and $L(Z(t))$ is independent of t for $t \geq 0$.*

Theorem 4 *Suppose that L is a lyapunov functional on $\mathcal{O}$ and that $\mathcal{P} \subset \mathcal{O}$ is a closed bounded set satisfying*

i. *For each $t \geq 0$, $\Phi_t(\mathcal{P}) \subset \mathcal{P}$ where Φ_t is the flow of the differential equation.*

ii. *X^* is the only orbit in $\mathcal{P}$ along which L is constant for $t \geq 0$.*

Then every orbit starting in $\mathcal{P}$ converges to X^* as $t \to \infty$.

Equivalently, the basin of attraction of X^* contains $\mathcal{P}$. In many examples the set $\mathcal{P}$ is of the form $\{L \leq \alpha\}$.

Theorem 4 and Theorem 2 follow quickly from Theorem 3