The Multiple Roots Algorithm

The input is an $n \times n$ complex matrix A and an eigenvalue, λ of A. Denote by $m \geq 1$ the multiplicity of λ as a root of the characteristic polynomial

$$p(z) \ := \ \det\big(zI - A\big)\,.$$

The output of the algorithm is m linearly independent solutions,

$$\Phi_j(t), \qquad 1 \leq j \leq m\,.$$

of the differential equation $X' \ = \ A\,X$.

Remarks. i. Taking the outputs from the distinct roots of p yields n linearly independent solutions.

ii. Their linear combinations give the general solution.

iii. Using them as columns yields a fundamental matrix, $\Psi(t)$.

iv. The exponential is computed using $e^{At} = \Psi(t)\Psi^{-1}(0)$.

v. First check to see if there are m independent eigenvectors $\mathbf{v}_k$ for λ. In that case there are m independent solutions $e^{\lambda t}\,\mathbf{v}_k$.

Step I. Find vectors ξ_j $1 \leq j \leq m$ which form a basis of the m dimensional subspace

$$\text{Null}\Big(\big(A - \lambda I\big)^m\Big)\,.$$

This is called the **generalized eigenspace** associated to λ.

Step II. For $1 \leq j \leq m$ define the solutions

$$\Phi_j(t) \ := \ e^{\lambda t} \sum_{k=0}^{m-1} \frac{t^k}{k!}\,\big(A - \lambda I\big)^k \xi_j\,. \qquad \big(\text{Recall that } \ 0! \ := \ 1\big).$$

Remarks. i. In case of n distinct eigenvalues all the m are equal to 1 and this reduces to the standard eigenvalue eigenvector method.

ii. More generally, when there is a basis of eigenvectors, $(A - \lambda I)\xi_j = 0$ so the terms with $j \geq 1$ all vanish, one recovers the standard method. The additional terms are only required $N((A - \lambda I)^m)$ is *strictly* larger than $N(A - \lambda I)$. Equivalently, there is an eigenvalue of multiplicity $m > 1$ whose space of eigenvectors has dimension $< m$.

iii. That the Φ_j are solutions can be checked by differentiation using the fact that $(A - \lambda I)^m \xi_j = 0$.

iv. The difficult fact in this algorithm is that the nullspace in Step I has dimension equal to m.