Perturbations of Linear Sinks

Summary. *Small perturbations of linear sinks yield globally defined exponentially decreasing solutions. In particular an equilibrium whose linearization is asymptotically stable is itself asymptotically stable.*

1 The main result.

The following result which includes the possibility of nonautonomous perturbations has a simple elegant proof and several distinct applications. Denote by $\mathbb{V}$ a real or complex finite dimensional normed vector space with norm $\|\cdot\|$. The linear asymptotically stable dynamics is

$$X' \;=\; A\,X \tag{1.1}$$

where $A:\mathbb{V}\to\mathbb{V}$ is a linear transformation with eigenvalues with strictly negative real part so there exist strictly positive K and ρ so that for all $t\geq 0$,

$$\left\|e^{At}\right\| \;\leq\; K\,e^{-\rho t}\,. \tag{1.2}$$

Here and in the following the norm of a linear transformation is taken as

$$\|B\| \;:=\; \max_{\|X\|\leq 1}\|BX\|\,.$$

Consider the perturbed dynamics

$$X' \;=\; A\,X + f(t,X) \tag{1.3}$$

where f is continuous in t, X and uniformly Lipschitzean in X in the sense that there is a Λ so that for all t, X, Y, $\|f(t,x)-f(t,Y)\|\leq\Lambda\|X-Y\|$. This implies uniqueness and existence for $0\leq t<\infty$ for the initial value problem. The next result shows that if f is sufficiently small then (1.3) inherits the asymptotic stability of (1.1).

Theorem 1.1 *Suppose that there is an $\eta>0$ so that*

$$K\,\eta \;<\; \rho, \qquad \text{and} \qquad \forall t,x, \quad \|f(t,X)\| \;\leq\; \eta\,\|X\|\,. \tag{1.4}$$

Then there is a $\tilde\rho>0$ so that solutions of (1.3) satisfy for all $t\geq 0$,

$$\|X(t)\| \;\leq\; K\,e^{-\tilde\rho t}\,\|X(0)\|\,. \tag{1.5}$$

Proof. The variation of parameters formula implies that for $t \geq 0$

$$X(t) \;=\; e^{At} X(0) \;+\; \int_0^t e^{A(t-s)}\, f(s, X(s))\; ds\,.$$

Using estimates (1.2) (1.4) together with the triangle inequality for sums and for integrals implies that

$$\|X(t)\| \;\leq\; K\, e^{-\rho t}\, \|X(0)\| \;+\; \int_0^t K\, e^{-\rho(t-s)}\, \eta\, \|X(s)\|\; ds\,.$$

Multiply through by $e^{\rho t}$ to find

$$e^{\rho t}\, \|X(t)\| \;\leq\; K\, \|X(0)\| \;+\; K\, \eta \int_0^t e^{\rho s}\, \|X(s)\|\; ds\,.$$

Then $\phi(t) := e^{\rho t}\, \|X(t)\|$ satisfies,

$$\phi(t) \;\leq\; K\, \|X(0)\| \;+\; K\, \eta \int_0^t \phi(s)\; ds\,.$$

Gronwall's inequality implies that

$$\phi(t) \;\leq\; K\, \|X(0)\|\, e^{K\eta\, t}\,.$$

Multiply through by $e^{-\rho t}$ to find

$$\|X(t)\| \;\leq\; K\, \|X(0)\|\, e^{(K\eta - \rho) t}\,.$$

This proves the desired result with $\tilde{\rho} := \rho - K\eta > 0$ thanks to (1.4). $\square$

2 Application to linear equations.

Consider the linear equation

$$X' \;=\; A\, X \;+\; B(t)\, X\,, \tag{2.1}$$

with $B(t)$ a continuouos function valued in the linear transformations on $\mathbb{V}$.

Theorem 2.1 *Suppose that A, K, and ρ satisfy* (1.2)*. Suppose in addition that there is an $\eta > 0$ and $T \geq 0$ so that*

$$K\, \eta \;< \rho, \qquad \text{and} \qquad \forall t \geq T, \quad \|B(t)\| \;\leq\; \eta\,. \tag{2.2}$$

Then there are strictly positive constants $\tilde{K}$ *and* $\tilde{\rho}$ *so that solutions of* (2.1) *satisfy for all* $t \geq 0$

$$\|X(t)\| \;\leq\; \tilde{K}\, e^{-\tilde{\rho} t}\, \|X(0)\|\,. \tag{2.3}$$

Proof. Theorem 1.1 implies that for $t \geq T$ one has

$$\|X(t)\| \;\leq\; K\, e^{-\tilde{\rho}(t-T)}\, \|X(T)\|\,. \tag{2.4}$$

Let $\Psi(t)$ the fundamental matrix for (2.1) with $\Psi(0) = I$. By continuity define

$$M \;:=\; \max_{0 \leq t \leq T} \|\Psi(t)\| \;<\; \infty\,.$$

Then

$$\sup_{0 \leq t \leq T} \|X(t)\| \;\leq\; M\, \|X(0)\|\,. \tag{2.5}$$

Estimates (2.4) and (2.5) imply the desired estimate (2.3). □

3 Asymptotic stability of nonlinear equilibria.

Consider the nonlinear autonomous system

$$X' \;=\; F(X) \tag{3.1}$$

with equilibrium X_0. Suppose that F is a twice continuously differentiable function on the ball $\{\|X - X_0\| \leq R_1\}$.

Theorem 3.1 *If the linearization*

$$Y' \;=\; A\,Y\,, \qquad A \;:=\; D_X F(X_0)$$

has $Y = 0$ *as an asmyptotically stable equilibrium, then* X_0 *is an asymptotically stable equilibrium of* (3.1)*.*

We will prove a more precise stability estimate. It starts with some preparation. Translating coordinates we may suppose that $X_0 = 0$. Then Taylor's Theorem with remainder implies that

$$F(X) \;=\; A\,X \;+\; g(X) \qquad \text{on} \qquad \|X\| \leq R_1 \tag{3.2}$$

and there is a $C > 0$ so that g satisfies

$$\|g(X)\| \;\le\; C\,\|X\|^2 \qquad \text{on} \qquad \|X\| \le R_1\,.$$

Choose K and ρ so that (1.2) holds. Choose $0 < \eta$ so that $K\eta < \rho$ and choose $0 < R_2 \le R_1$ so that

$$C\,R_2 \;<\; \eta\,. \tag{3.3}$$

Then

$$\|g(X)\| \;\le\; \eta\,\|X\| \qquad \text{for} \qquad \|X\| \;\le\; R_1\,. \tag{3.4}$$

Theorem 3.2 *Suppose that R_1, R_2, C, η are as above. Then there is an $0 < R_3 \le R_2$ and strictly positive $\tilde{K}$ and $\tilde{\rho}$ so that solutions of* (3.1) *with $\|X(0)\| \le R_3$ exist for all $t \ge 0$, take values in $\|X\| \le R_1$, and satisfies*

$$\|X(t)\| \;\le\; \tilde{K}\,e^{-\tilde{\rho}t}\,\|X(0)\|\,. \tag{3.5}$$

Proof. The function g is only defined for $\|X\| \le R_1$. Extend it to all X so that for $\|X\| \ge R_1$, g is constant on rays through the origin. That is for $\|X\| \ge R_1$,

$$g(X) \;:=\; g\!\left(\frac{R_1\,X}{\|X\|}\right).$$

Then $\|g(X)\| \le \eta\|X\|$ for all X. (**Exercise.** *Verify this.*) Therefore Theorem 1.1 with $f(t,X) := g(X)$ implies that the equation

$$X' \;=\; A\,X \;+\; g(X) \tag{3.6}$$

is globally solvable and solutions satisfy (3.5).

Choose $0 < R_3 \le R_2$ so that $\tilde{K}R_3 \le R_1$. Then if $\|X(0)\| \le R_3$, (3.5) implies that for $t \ge 0$, $\|X(t)\| \le R_1$. Equations (3.2) and (3.6) imply that (3.1) holds.

$X(t)$ is therefore the unique solution of (3.1) with initial value $X(0)$. Therefore that solution takes values in $\|X\| \le R_1$ and satisfies (3.5). □