Parametric Resonance

Floquet Theory concerns linear homogeneous systems with periodic coefficients. Denote by $T > 0$ the period. The systems have the form

$$X'(t) \;=\; A(t)\,X(t)\,, \qquad A(t+T) = A(t)\,. \tag{0.1}$$

A is a continuous matrix valued function of t.

The phenomenon of parametric resonance is the following. It can happen that each of the frozen matrices $A(\underline{t})$ has spectrum in the left half plane so generates an asymptotically stable motion while the equation (1.1) has exponentially growing solutions. One cannot determine the spectrum of $\Phi(T)$ from the spectra of the $A(t)$.

At fist glance this seems unlikely. Each of the frozen dynamics is dissipative. It is surprising that when you do them one after the other they can generate growth. In this brief note I present a simple example which shows that the phenomenon is not hard to understand.

Begin with the spring equation $x'' + 4x = 0$ with general solution $a \sin 2t + b \cos 2t$. For $X = (x, x')$ the system is

$$X' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} X \;:=\; B\,X\,.$$

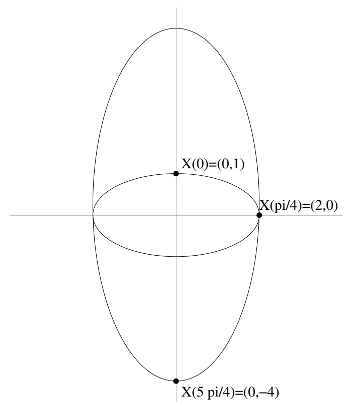
The solutions have least period is π. The physical energy is proportional to $X_1^2 + 4X_2^2$ a quantity that is constant on orbits. In the X-plane, orbits are ellipses twice as long on the X_1 axis as on the X_2-axis. The motion is clockwise. The orbit starting at $(0,1)$ reaches $(2,0)$ after $\pi/4$ units of time.

The spring $4x'' + x = 0$ with general solution $a \sin t/2 + b \cos t/2$ yields the system

$$X' = \begin{pmatrix} 0 & 1 \\ -1/4 & 0 \end{pmatrix} X \;:=\; C\,X\,.$$

The solutions have least period 4π. The orbits are ellipses twice as long along the X_2 axis as the X_1-axis traversed clockwise with period π. The point $(2,0)$ goes to $(0,-4)$ after π units of time.

Let $A(t)$ be the function so that $A(t) = B$ for $0 \le t \le \pi/4$ and $A(t) = C$ for $\pi/4 < t < 5\pi/4$. The function is extended to be periodic with period $5\pi/4$.



The above computation shows that $\Phi(5\pi/4)(0,1) = (0,-4)$. The number -4 is a Floquet multiplier. There are exponentially growing solutions. Since the trace of $A(t)$ vanishes for all t it follows that $\det \Phi(t) = 1$ for all t so the second Floquet multiplier is equal to $-1/4$ and there is an exponentially decaying solutions. Since $A(\underline{t})$ generate centers both phenomena are surprising.

Replacing $A(t)$ by $A(t) - \epsilon I$ yields an example with floquet multipliers $-4e^{-\epsilon\pi/2}$ and $(-1/4)e^{-\epsilon\pi/2}$. For ϵ small, they are close to -4 and $-1/4$.

To make an example with continuous $A(t)$, smoothly cutoff A so as to vanish on an interval of width $0 < \delta << 1$ about the jump discontinuities. Then $\Phi(5\pi/4)$ changes by a quantity $\sim \delta$. The new Φ has eigenvalues differing from the old ones by $\sim \delta$. Therefore, for δ small there is still an eigenvalue close to -4.