The Steps of Perturbation Theory

Abstract. The differentiable dependence on parameters in the fundamental existence and uniqueness theorem for ordinary differential equations justifies the computations of perturbation theory. This handout details the steps in perturbation computations.

Suppose that $y(t,\epsilon)$ is the solution of an ordinary differential equation in which the equation and the initial data depend smoothly on a parameter ϵ.

Goal. Compute the coefficients in the Taylor polynomials

$$y(t,\epsilon) \;\approx\; y(t,0) \;+\; \frac{\partial y(t,0)}{1!\,\partial\epsilon}\,\epsilon \;+\; \cdots \;+\; \frac{\partial^n y(t,0)}{n!\,\partial\epsilon^n}\,\epsilon^n \;:=\; y_0(t) + \epsilon\, y_1(t) + \frac{\epsilon^2}{2!} y_2(t) + \frac{\epsilon^n}{n!} y_n(t)\,.$$

Step I. Write the equation and the initial conditions paying special attention to note where the dependence on the parameter ϵ appears.

Step II. Unperturbed solution. Set $\epsilon = 0$ in the equations and the initial conditions to derive an initial value problem which determines the unperturbed solution

$$y(t,\epsilon)\big|_{\epsilon=0} \;=\; y(t,0) \;:=\; y_0(t)\,.$$

Solve for $y_0(t)$ if possible.

Step III. Differentiate wrt epsilon. Differentiate the equation and the initial condition with respect to ϵ, that is apply $\partial/\partial\epsilon$ to them.
This generates initial value problems for $\partial y/\partial\epsilon$ for all t,ϵ. Keep this equations as they are needed for possible higher order terms.

Step IV. Order epsilon term. Set $\epsilon = 0$ to generate an equation for

$$\frac{\partial y(t,\epsilon)}{\partial\epsilon}\bigg|_{\epsilon=0} \;=\; \frac{\partial y(t,0)}{\partial\epsilon} \;:=\; y_1(t)\,.$$

Solve for $y_1(t)$ if possible.
In the simplest cases, $y_0(t) + \epsilon y_1(t)$ is the desired approximate solution. If $y_1(t)$ is identically equal to zero, then you need to compute more terms to assess the leading order effect of the perturbations.

Higher order terms V. If more terms are needed to obtain the desired information continue as follows.
If the equations for $\partial^j y(t,\epsilon)/\partial\epsilon$ have just been treated, differentiate the equations and initial conditions with respect to ϵ. Set $\epsilon = 0$ to determine an initial value problem determining

$$\frac{\partial^{j+1} y(t,\epsilon)}{\partial\epsilon^{j+1}}\bigg|_{\epsilon=0} \;=\; \frac{\partial^{j+1} y(t,0)}{\partial\epsilon^{j+1}} \;:=\; y_{j+1}(t)\,.$$

Solve for $y_{j+1}(t)$ if possible.

Example. Find an approximate solution for $0 < \epsilon << 1$ of the weakly nonlinear initial value problem,

$$x' \;=\; x - \epsilon\, x^2, \qquad x|_{t=0} \;=\; 1\,. \tag{1}$$

Denote by $x(t,\epsilon)$ the solution. The fundamental existence and uniqueness theorem implies that x is a smooth function of t, x so an approximation is given by Taylor's Theorem,

$$x(t,\epsilon) \;\approx\; x(t,0) \;+\; \epsilon\frac{\partial x(t,0)}{\partial\epsilon}\,.$$

We compute $\partial_\epsilon x(t,0)$.
First differential (1) with respect to ϵ to find

$$\frac{\partial}{\partial \epsilon}\frac{\partial}{\partial t}x \;=\; \frac{\partial}{\partial \epsilon}\Big(x \;-\; \epsilon\, x^2\Big)\,, \qquad \frac{\partial x(0,\epsilon)}{\partial \epsilon} \;=\; 0\,.$$

Using equality of mixed partials and evaluating the derivative on the right yields

$$\frac{\partial}{\partial t}\,\frac{\partial x(t,\epsilon)}{\partial \epsilon} \;=\; \frac{\partial x(t,\epsilon)}{\partial \epsilon} \;-\; x^2 \;-\; 2\,\epsilon\, x\,\frac{\partial x(t,\epsilon)}{\partial \epsilon}\,.$$

The unperturbed solution $x(t,0) = e^t$. Define $z(t) := \partial x(t,0)/\partial\epsilon$. Since $x^2 = e^{2t}$ one finds the initial value problem

$$z' \;=\; z \;-\; e^{2t}\,, \qquad z(0) \;=\; 0\,.$$

There is one solution of the differential equation of the form Ae^{2t}. Plugging in yields $2A = A - 1$ whence $A = -1$. The general solution of the differential equation for z is therefore

$$C\,e^t \;-\; e^{2t}\,.$$

The initial condition yields $C = 1$ whence

$$z \;=\; -e^{2t} + e^t\,, \qquad x \;\approx\; e^t \;+\; \epsilon\big(-e^{2t} + e^t\big)\,.$$