Phase Planes for Two by Two Linear Systems

1 Introduction.

Consider linear constant coefficient systems

$$X' = AX, \tag{1.1}$$

where A is a real matrix with distinct non zero eigenvalues. This is an open dense subset of the set of four dimensional set of 2×2 real matrices. The complementary set is a closed set of dimension three, hence consists of rare occurences defined by,

$$\det A = 0, \qquad \text{or,} \qquad \text{discrimnant} = 0.$$

The condition of distinct nonzero eigenvalues is stable under small perturbations of A which is another way of saying that the set is open. In class we proved the the complementary set is closed and of four dimensional volume equal to zero. The systems studied here are **generic**.

2 Change of variables

A sub theme in these computations is the behavior of a linear constant coefficient system

$$X' = AX$$

when one makes a linear change of variable $T\alpha = X$ where T is an invertible matrix. This computation is valid for general $N \times N$ systems in which case the matrix T is also $N \times N$. It is found in §3.4 of HSD.

Writing $X(t) = T\alpha(t)$, the differential equation for the $\alpha(t)$ is

$$\big(T\alpha\big)' = AT\alpha.$$

The left hand side is equal to $T\alpha'$ so multiplying on the left by T^{-1} yields

$$\alpha' = T^{-1}AT\alpha. \tag{2.1}$$

The new variables satisfy a system of the same form with the matrix A changed by a similarity to $T^{-1}AT$.

If you understand the $T^{-1}AT$ differential equation then you understand the original problem. The strategy is to make $T^{-1}AT$ as simple as possible so the transformed problem is easier. It is also true that $T^{-1}AT$ is the matrix of A in the new variables Y defined by $X = TY$, equivalently in the new basis whose elements are the columns of the matrix T. In linear algebra $A \mapsto T^{-1}AT$ is called a similarity transformation and the problem of equivalence of matrices under similarity is often discussed.

Exercise 2.1 *Prove that if two matrices A and B have N distinct eigenvalues then they are similar if and only if they have the same eigenvalues.*

3 Saddles. Real eigenvalues of opposite signs.

If A has real eigenvalues of opposite signs $\lambda_- < 0 < \lambda_+$ then choosing nonzero real eigenvectors v_- and v_+ the general solution is

$$X(t) \;=\; \alpha_- e^{\lambda_1 t}\, v_- \;+\; \alpha_+ e^{\lambda_+ t}\, v_+\,.$$

The line $\mathbb{R}v_+$ is invariant under the flow. The flow at time t simply multiplies by $e^{\lambda_+ t}$ so points move outward exponentially. As $t \to -\infty$ they converge exponentially to 0. This line is called the *unstable manifold.*

The line $\mathbb{R}v_-$ is also invariant and the flow is multiplication by $e^{\lambda_- t}$ so contracts exponentially toward the origin. As $t \to -\infty$ the flow expands exponentially toward infity. This line is called the *stable manifold.*

Since $Av_+ = \lambda_+ v_+$, the vector field on $\mathbb{R}v_+$ points parallel to the line and away from the origin. The length is proportional to the distance from the origin.

The vector fields along $\mathbb{R}v_-$ is similar except pointing inward.

As $t \to \infty$ the $c_+ e^{\lambda_+ t}$ term is dominant and integral curves are asymptotic to the line $\mathbb{R}v_+$.

As $t \to -\infty$ integral curves are asymptotic to the line $\mathbb{R}v_-$.

An integral curve off the stable and unstable manifolds comes in from the direction of $\mathbb{R}v_-$ turns and leaves approaching $\mathbb{R}v_+$ as $t \to \infty$. They have an aspect that is roughly hyperbolic.

To see that they are NOT hyperbolas when the modulus of the λ's are not equal note that the approach to the asymptote correponding to the eigenvalue with smaller modulus is more rapid than the approach to the other since the decay of the negligible term is more rapid.

Consider the example

$$X' \;=\; \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} X\,.$$

The general solution is

$$x_1 = c_1 e^{\lambda_- t}\,, \qquad x_2 = c_2 e^{\lambda_+ t}\,.$$

Therefore on orbits one has

$$|x_1|^{\lambda_+}/|x_2|^{\lambda_-} \;=\; \text{independent of time.}$$

Since $\lambda_- < 0$, $|x_1|^{\lambda_+}/|x_2|^{\lambda_-} = |x_1|^{\lambda_+}|x_2|^{|\lambda_-|}$. Therefore the continuous function

$$\varphi(x_1, x_2) \;:=\; |x_1|^{\lambda_+}|x_2|^{|\lambda_-|} \tag{3.1}$$

is constant on orbits. When $|\lambda_+| = |\lambda_-|$ this shows that the orbits are hyperbolas. And when the lambdas are not of equal magnitude the orbits are not hyperbolas.

Next show by a change of basis that there is always a continuous quantity like (3.1) that is constant on orbits. Introduce the components $v_+ = (v_{+,1}\,,\, v_{+,2})$ and the matrix

$$\begin{pmatrix} v_{-,1} & v_{+,1} \\ v_{-,2} & v_{+,2} \end{pmatrix}$$

whose first (resp. second) column is v_- (resp. v_+). The columns being linearly independent this is a nonsingular matrix so we can define

$$T \;:=\; \begin{pmatrix} v_{-,1} & v_{+,1} \\ v_{-,2} & v_{+,2} \end{pmatrix}, \qquad T^{-1} \;:=\; \begin{pmatrix} v_{-,1} & v_{+,1} \\ v_{-,2} & v_{+,2} \end{pmatrix}^{-1}.$$

Then

$$T(1,0) \;=\; v_-, \qquad T(0,1) \;=\; v_+\,, \qquad T^{-1}v_- \;=\; (1,0), \qquad T^{-1}v_+ \;=\; (0,1)\,, \tag{3.2}$$

where the last two follow from the first two upon multiplying by T^{-1}.

For $X \in \mathbb{R}^2$ denote by $(\alpha_-, \alpha_+) := \alpha$ the coordinates of X in the basis v_-, v_+ that is

$$X \;=\; \alpha_1\, v_- \;+\; \alpha_+\, v_+ \;=\; \begin{pmatrix} v_{-,1} & v_{+,1} \\ v_{-,2} & v_{+,2} \end{pmatrix} \alpha \;=\; T\,\alpha\,, \qquad \alpha = T^{-1}X\,.$$

Equation (3.2) shows that

$$T^{-1}\,A\,T(1,0) \;=\; T^{-1}\,A\,v_- \;=\; T^{-1}\,\lambda_- v_- \;=\; \lambda_-(1,0)\,.$$

Exercise 3.1 *Show that* $T^{-1}AT(0,1) = \lambda_+(0,1)$*, and,*

$$T^{-1}AT \;=\; \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}.$$

The differential equation satisfied by the coordinates α is

$$\alpha' \;=\; \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} \alpha\,, \qquad \alpha'_- = \lambda_-\alpha_-\,, \qquad \alpha'_+ = \lambda_+\alpha_+\,.$$

This is the example analysed earlier so we know that $\varphi(\alpha) = \varphi(T^{-1}X)$ is a continuous *conserved quantity*. Also called an *integral of motion*.

The figure on page 92 of Brauer and Nohel and page 41 of Hirsch-Smale-Devaney are exactly for the exceptional case of $\lambda_- = -\lambda_+$ with hyperbolic orbits. The reader is encouraged to plot (using `pplane` in Matlab) an example where this condition is violated to see the symmetry breaking which is typical.

In terms of the original coordinates and the matrix elements T^{-1}_{ij} one has

$$\alpha_- \;=\; T^{-1}_{11}x_1 + T^{-1}_{12}x_2, \qquad \alpha_+ \;=\; T^{-1}_{21}x_1 + T^{-1}_{22}x_2,$$

so a conserved quantity is

$$\varphi(\alpha) \;=\; |\alpha_1|^{\lambda_+}|\alpha_2^{|\lambda_-|} \;=\; |T^{-1}_{11}x_1 + T^{-1}_{12}x_2|^{\lambda_+}\;|T^{-1}_{21}x_1 + T^{-1}_{22}x_2|^{|\lambda_-|}\,.$$

Summary. *There are two invariant lines, in the directons of the eigenvectors. The flow is outward (resp. inward) in the direction of the eigenvector with positive (resp. negative) eigenvalues. The equilibrium is unstable. The other integral curves are asymptotic to the to these two lines and resemble hyperbolas, but lacking their symmetry and quadric equation (unless* $\lambda_- = -\lambda_+$*). There is a nontrivial continuous integral of motion.*

4 Improper nodes. Distinct real roots of the same sign.

Consider the case of positive roots $0 < \lambda_1 < \lambda_2$. The general solution is

$$c_1\, e^{\lambda_1 t}\, v_1 \;+\; c_2\, e^{\lambda_2 t}\, v_2\,.$$

The lines in the directions of v_j are invariant and the flow is outward on both. The invariant lines correspond to one or the other of c_1, c_2 vanishing. The expansion on the v_2 line is more rapid so that orbits tend in the limit $t \to \infty$ to be nearly parallel to v_2. For $t \to -\infty$ the $e^{\lambda_1 t}$ term is dominant and the orbits approach the origin tangent to the v_1 line.

Introducing the basis v_1, v_2 and corresponding coordinates α_1, α_2 and matrices T^{-1}, T as in the preceding section one has

$$T^{-1}AT \;=\; \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \alpha \;=\; \big(c_1 e^{\lambda_1 t}\,,\, c_2 e^{\lambda_2 t}\big)\,.$$

The quantity

$$\frac{|\alpha_2|^{\lambda_1}}{|\alpha_1|^{\lambda_2}} \;=\; \frac{c_2^{\lambda_1}\; e^{\lambda_1\lambda_2 t}}{c_1^{\lambda_2}\; e^{\lambda_1\lambda_2 t}} \;=\; \frac{c_2^{\lambda_1}}{c_1^{\lambda_2}}$$

is constant on orbits. Since $\lambda_2 > 0$ this is discontinous along the line $\alpha_1 = 0$.

The orbits in α coordinates are as in figure 2.9 of Brauer and Nohel and figure 3.3.b of Hirsch-Smale-Devaney. The orbits in X space are their image by the linear transformation T^{-1}. Qualitatively they look like figure 3.7 in HSD with arrows reversed.

Orbits move away from the origin growing infinitely large. The origin is a *source* or *repellor*.

Proposition 4.1 *If the eigenvalues of A both have strictly positive (resp. negative) real part, then the only continuous conserved quantities are constants.*

Proof. We treat the case of postive real part. Suppose that φ is a continuous conserved quantity. If P is any point, Denote by $X(t)$ the orbit with $X(0) = P$. Then for all t,

$$\varphi(P) \;=\; \varphi(X(0)) \;=\; \varphi(X(t))\,.$$

As $t \to -\infty$, $X(t) \to 0$. By continuity of φ at the origin,

$$\varphi(0) \;=\; \lim_{t\to\infty} \varphi\big(X(t)\big) \;=\; \varphi\Big(\lim_{t\to-\infty} X(t)\Big) \;=\; \varphi(P).$$

Therefore φ is constant. ∎

The case of distinct negative real eigenvalues is analogous to the case of postive distinct eigenvalues generating a *sink* which is an *attractor*. The negative case behaves as the positive case with time reversed.

Summary. *There are two invariant lines. For positive real part, the flow is outgoing on both. The equilibrium is unstable. The line corresponding to the larger eigenvalue dominates for t large positive, while the other dominates for $t \to -\infty$. There are no nonconstant continuous conserved quantitites.*

5 Centers and spirals. Complex conjugate eigenvalues.

Solutions are generated by

$$e^{\lambda t} v, \qquad e^{\overline{\lambda} t}\overline{v}\,.$$

Equivalently, by the real and imaginary parts of $e^{\lambda t}v$. Write

$$\lambda = a + ib, \qquad v = r + is,$$

in terms of their real and imaginary parts. Then

$$e^{\lambda t}v \;=\; e^{at}\, e^{ibt}\, v\,.$$

The term $e^{ibt}v$ is of constant magnitude so there is exponential growth (resp. exponential decay) when $a > 0$ (resp. $a < 0$). The analysis is most conveniently done by considering first the case $a = 0$.

An exponential in time change of variable reduces to the case $a = 0$. Indeed, the function $X(t)$ satisfies $X' = AX$ if an only if $Y(t) = e^{-at}X(t)$ satisfies $Y' = (A - aI)Y$. Therefore is $X' = AX$ has spiral solutions, then $Y' = (A - (\mathrm{tr}\, A)/2)I)Y$ is the associated system that has $a = 0$ so is a center. The spirals are exactly the solution of the system with a center times the purely exponential factor $e^{t\,\mathrm{tr}\,A/2}$.

Example 5.1 *For*

$$A \;=\; \begin{pmatrix} 3 & 5 \\ -2 & -2 \end{pmatrix},$$

$\mathrm{tr}\, A = 3 - 2 = 1$, *so*

$$A - (\mathrm{tr}\, A/2)I \;=\; \begin{pmatrix} 2.5 & 5 \\ -2 & -2.5 \end{pmatrix}$$

is the trace free matrix whose motion is a center. It is further analyzed in the Ellipse Axes Handout.

5.1 Analysis of Centers. Eigenvalues $0 \neq \pm bi$.

The solution $e^{\lambda t}v$ is,

$$\big(\cos bt + i \sin bt\big)(r + is) \;=\; \big(r\, \cos bt - s\, \sin bt\big) \;+\; i\big(r\, \sin bt + s\, \cos bt\big)\,.$$

Real solutions are generated by the real and imaginary parts,

$$r\, \cos bt - s\, \sin bt\,, \qquad r\, \sin bt + s\, \cos bt\,.$$

Introduce the basis r, s and corresponding coordinates and the notation T following the case of saddles,

$$X \;=\; \alpha_1 r \;+\; \alpha_2 s \;=\; \begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \end{pmatrix} \alpha \;:=\; T\alpha\,, \qquad \alpha = T^{-1}X\,.$$

Therefore

$$T(1,0) = r, \qquad T(0,1) = s\,. \qquad (1,0) = T^{-1}r, \qquad (0,1) = T^{-1}s, \tag{5.1}$$

Compute

$$T^{-1}AT(1,0) = T^{-1}Ar\,. \tag{5.2}$$

Use

$$A(r + is) = ib((r + is), \qquad \text{equivalently}, \qquad Ar + iAs = -bs + ibr\,.$$

Taking the real and imaginary parts yield

$$Ar = -bs\,, \qquad \text{and,} \qquad As = br\,.$$

Continuing with (5.2) and using (5.1) yields

$$T^{-1}AT(1,0) = T^{-1}Ar = T^{-1}(-bs) = -b(0,1)\,.$$

Exercise 5.1 *Show that $T^{-1}AT(0,1) = b(1,0)$, and,*

$$T^{-1}AT = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\,. \tag{5.3}$$

The equations for the α coordinates are

$$\alpha' \;=\; \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\alpha\,, \qquad \alpha_1' \;=\; b\,\alpha_2, \qquad \alpha_2' \;=\; -b\,\alpha_1,\,. \tag{5.4}$$

Multiply the first equation in (5.4) by α_1, the second by α_2, and add to find that

$$\alpha_1\alpha_1' \;+\; \alpha_2\alpha_2' \;=\; \alpha_1\,b\,\alpha_2 \;-\; \alpha_2\,b\,\alpha_1 \;=\; 0\,. \qquad \textit{This is cool and clever.}$$

This shows that $\alpha_1^2 + \alpha_2^2$ has vanishing derivative so is constant on orbits. Therefore the orbits are circles in the α coordinates. Since $\|\alpha\|$ is constant on orbits and $\|\alpha' = |b|\|\alpha$ the derivative is also constant. The orbits are circles of radius r traversed as speed $|b|r$. See figure 2.11 in Brauer and Nohel and 3.4 in Hirsch-Smale-Devaney. The conserved quantity in X coordinates is computed using

$$\alpha_1 \;=\; T^{-1}_{11}x_1 + T^{-1}_{12}x_2\,, \qquad \alpha_2 \;=\; T^{-1}_{21}x_1 + T^{-1}_{22}x_2\,,$$

to be

$$\big(T^{-1}_{11}x_1 + T^{-1}_{12}x_2\big)^2 \;+\; \big(T^{-1}_{21}x_1 + T^{-1}_{22}x_2\big)^2\,.$$

Its level sets are bounded conic sections, hence ellipses. The orbits in the X coordinates are ellipses. In a separate handout we address the question of computing the principal axes, eccentricity, and direction of rotation for the elliptical orbits.

Summary. *For nonzero purely imaginary eigenvalues, the orbits are ellipses. There is a positive definite quadratic conserved quantity. The origin is a stable equilibrium. Under small real perturbations of A, the eigenvalues will typically leave the imaginary axis remaining a complex conjugate pair*

5.2 Spirals. Eigenvalues $a \pm ib$, $a \neq 0 \neq b$.

The solutions are exactly as in the preceding section just multiplied by e^{at}. For $a > 0$ the orbits are ellipses amplified by an exponentially growing factor. They spiral out. For $a < 0$ they spiral in. The orbits in α coordinates are given in figure 2.10 of Brauer and Nohel and fig 3.5 of Hirsch-Smale-Devaney.

Proposition 4.1. shows that there are no non constant continuous conserved quantities.

Summary *For $a > 0$ the orbits are elliptical spirals growing exponentially called a spiral source. For $a < 0$ they are elliptical spirals shrinking exponenetially, a spiral sink. There are no nonconstant continuous integrals of motion.*

Example 5.2 The vibrating spring with spring constant k has equation $x'' + k\,x = 0$. The restoring force is $k > 0$ times the displacement x from equilibrium. If the force is repulsive the equation is $x'' - k\,x = 0$. Conservation of energy in both cases yields the continuous conserved quantity

$$\frac{\dot{x}^2}{2} \pm \frac{k\,x^2}{2}.$$

For the plus sign the phase plane is a center with elliptical obits. For the minus sign the orbits are hyperbolas, the eigenvalues in that case being $\pm k^{1/2}$ so of equal magnitude. Saddles and centers are the only cases with conserved quantities and these basic example fall in those classes. For generic saddles the opposite sign eigenvalues will not be of equal amplitude and the orbits will not be hyperbolas.

Exercise 5.2 *Show that for any real $a \neq 0$*

$$x'' + a\,x' + k\,x = 0$$

yields a saddle. Show that the eigenvalues of opposite sign have equal amplitude if and only if $a = 0$. **Discussion.** The orbits are hyperbolas and the conserved quantity quadratic if and only if $a = 0$.