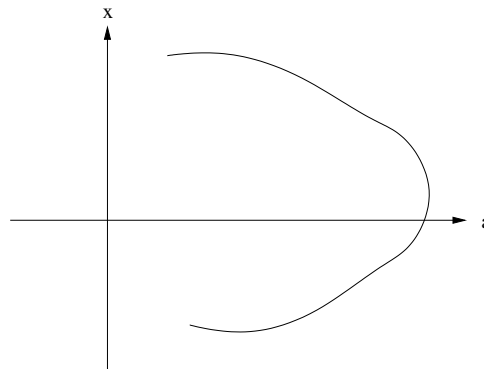


Bifurcation Theory

Summary. The first three sections treat problems in dimension 1. A final section shows that for $N > 1$ a reduction to the scalar case is possible. The Hopf-Andreev bifurcation is special to $N > 1$. The main tool is the Implicit Function Theorem. This is an excellent opportunity to learn that Theorem.

1 The single bifurcation curve case

This section title looks like a nonsequitor since bifurcation is about more than one solution. That is what the "bi" indicates. However, a curve in (a, x) -space which doubles back on itself can have more than one point for a given parameter value a .



Study equilibria of $x' = f(a, x)$ and their dependence on a . The equilibria are the zero set

$$\{(a, x) : f(a, x) = 0\}.$$

Suppose that $(\underline{a}, \underline{x})$ is an equilibrium, that is

$$f(\underline{a}, \underline{x}) = 0. \quad (1.1)$$

The *Implicit Function Theorem* asserts the following things.

i. If

$$\nabla_{a,x} f(\underline{a}, \underline{x}) \neq 0 \quad (1.2)$$

then near $(\underline{a}, \underline{x})$ the zero set is a smooth curve in a, x -space.

ii. The normal vectors to the curve are parallel to $\nabla_{a,x} f(a, x)$.

iii. The curve is locally a graph $x = k(a)$ near \underline{a} when $f_x(\underline{a}, \underline{x}) \neq 0$. It is locally a graph $a = h(x)$ when $f_a(\underline{a}, \underline{x}) \neq 0$.

The derivatives of h or k can be computed by implicit differentiation.

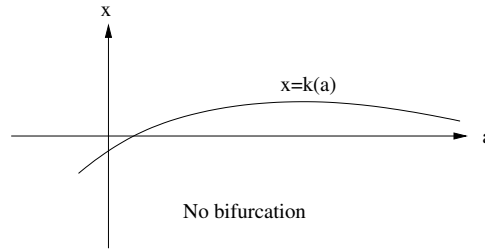
1.1 The no bifurcation case

If

$$\frac{\partial f(\underline{a}, \underline{x})}{\partial x} \neq 0 \quad (1.3)$$

the Implicit Function Theorem implies that $\{f = 0\}$ is locally a graph

$$x = k(a), \quad k(\underline{a}) = \underline{x}.$$



For each $a \approx \underline{a}$ there is one equilibrium $x = k(a)$ so there is no bifurcation.

Differentiate $f(a, k(a)) = 0$ with respect to a to find $f_a + f_x k' = 0$. The slope is given by

$$k'(a) = \frac{-f_a(a, k(a))}{f_x(a, k(a))}. \quad (1.4)$$

Next compute the stability of the equilibria so long as (1.3) holds, The linearized equation at the equilibrium $x = k(a)$ is $y' = f_x(a, k(a))y$. The coefficient, $f_x(a, k(a))$ is not equal to zero. The equilibrium \underline{x} of $x' = f(\underline{a}, x)$ is asymptotically stable when $f_x(\underline{a}, \underline{x}) < 0$. In the opposite case $f_x(\underline{a}, \underline{x}) > 0$, orbits converge to \underline{x} as $t \rightarrow -\infty$. Since $f_x(\underline{a}, \underline{x}) \neq 0$ its sign does not change. The nearby equilibria remain either attracting or repelling. They can change stability only if f_x vanishes.

1.2 The bifurcation case

A more interesting case is when (1.2) holds and

$$f_x(\underline{a}, \underline{x}) = 0. \quad (1.5)$$

Then $f_a(\underline{a}, \underline{x}) \neq 0$ so the Implicit Function Theorem implies that there is a curve of equilibrium given by

$$a = h(x), \quad \text{with} \quad h(\underline{x}) = \underline{a}, \quad h'(\underline{x}) = 0, \quad h'' = \frac{-f_{xx}(\underline{a}, \underline{x})}{f_a(\underline{a}, \underline{x})}. \quad (1.6)$$

To compute the last formulas, differentiate $f(h(x), x) = 0$ with respect to x to find

$$f_a(h(x), x) h'(x) + f_x(h(x), x) = 0.$$

implying $h'(\underline{x}) = 0$. Differentiate again with respect to x to find,

$$(f_{aa} h' + f_{ax}) h' + f_a h'' + f_{xa} h' + f_{xx} = 0.$$

At $x = \underline{x}$, $h'(\underline{x}) = 0$ eliminates three of the five terms. This yields the formula for $h''(\underline{x})$.

The figure at the start of §1 is such a curve with $h' = 0$ at the point with the vertical tangent and $h'' < 0$ at that point so the curve lies (locally) to the left of this vertical tangent. This is a **subcritical** bifurcation since the equilibria exists for parameter values a smaller than \underline{a} .

Exercise 1.1 Suppose that the hypotheses of this section hold and that $h''(\underline{x}) \neq 0$. Show that for $x \approx \underline{x}$ the equilibria on $\{f = 0\}$ have opposite stabilities for $x > \underline{x}$ and $x < \underline{x}$. **Hint.** Show that the leading order of the Taylor expansion is $f_x(h(x), x) \approx f_{xx}(\underline{a}, \underline{x})(x - \underline{x})$. **Discussion.** At the bifurcation point a stable and unstable branch meet and annihilate.

2 Two curves crossing

In order to have a more complicated zero set near $\underline{a}, \underline{x}$, one must have $\nabla_{a,x} f(\underline{a}, \underline{x}) = 0$. Then the leading order Taylor expansion of f near $\underline{a}, \underline{x}$ is

$$f(a, \underline{x}) = \left\langle \begin{pmatrix} f_{aa}(\underline{a}, \underline{x}) & f_{ax}(\underline{a}, \underline{x}) \\ f_{ax}(\underline{a}, \underline{x}) & f_{xx}(\underline{a}, \underline{x}) \end{pmatrix} \begin{pmatrix} a - \underline{a} \\ x - \underline{x} \end{pmatrix}, \begin{pmatrix} a - \underline{a} \\ x - \underline{x} \end{pmatrix} \right\rangle + \text{higher order terms}.$$

The matrix of second derivatives of f is real and symmetric. It therefore has two real eigenvalues, possibly equal. Since singular matrices are rare it is expected that neither of the two eigenvalues is equal to zero.

If both are positive then f has a strict local minimum at $\underline{a}, \underline{x}$ so $f > 0$ on a punctured neighborhood of $\underline{a}, \underline{x}$ and the equilibrium is isolated. Similarly the equilibrium is isolated if the matrix has two negative eigenvalues.

The remaining expected case is when the matrix of second derivatives has one positive and one negative real eigenvalue. Then the graph $z = f(a, x)$ near $\underline{a}, \underline{x}$ is saddle shaped and the level set at height $z = 0$ consists of two curves crossing transversally at $\underline{a}, \underline{x}$. The zero set is a curvy X shaped figure. This shows that after a single smooth curve the next expected behavior for the set $\{f = 0\}$ is an X shaped crossing.

Example 2.1 The function $f(a, x) = x(x-a)$ has zero set that is the union of the lines $\{x = 0\}$ and $\{x = a\}$ that cross at the origin.

Continuing with the general situation, translate coordinates so that $\underline{a}, \underline{x}$ is the origin and introduce an eigenbasis v_j for the matrix of second derivatives and corresponding coordinates

$$X = \alpha_1 v_1 + \alpha_2 v_2.$$

Suppose that the eigenvalues satisfy $\lambda_1 < 0 < \lambda_2$. Then to leading order the zero set is given by

$$\lambda_2 \alpha_1^2 + \lambda_1 \alpha_2^2 = 0, \quad \alpha_1 = \pm \sqrt{\frac{\lambda_2}{\lambda_1}} \alpha_2.$$

Note that though the eigendirections of the matrix of second derivatives are orthogonal, the two lines from the zero set are usually not orthogonal.

Exercise 2.1 Find necessary and sufficient conditions on the eigenvalues so that the two curves of the X cross at a right angle.

3 Transcritical and pitchfork

The analysis of the X shaped crossings is easiest when one of the branches is known. There are many applications, where $x = \underline{x}$ is an equilibrium for all values of a . The study of the X shape in this case reduces to studying the second branch near $(\underline{a}, \underline{x})$. We analyze that branch.

Translating x coordinates we may suppose that $\underline{x} = 0$. Seek a second curve of equilibria through $(\underline{a}, \underline{x}) = (\underline{a}, 0)$. We are given that

$$f(a, 0) = 0 \quad \text{for all } a. \quad (3.1)$$

In order for there to be two branches intersecting at $(\underline{a}, \underline{x})$ is it necessary that (1.2) be violated, that is

$$\nabla_{a,x} f(\underline{a}, \underline{x}) = 0. \quad (3.2)$$

That $f_a = 0$ follows from (3.1). *The second condition*

$$f_x(\underline{a}, \underline{x}) = 0 \quad (3.3)$$

is a necessary condition for \underline{a} to be the intersection point of two curves of equilibria.

To analyze further, the strategy is to separate out the root $x = 0$ using the identity

$$f(a, x) = x g(a, x), \quad g(a, x) := \int_0^1 \frac{\partial f(a, \theta x)}{\partial x} d\theta. \quad (3.4)$$

This special case of Taylor's Theorem follows from the Fundamental Theorem of Calculus applied to

$$\kappa(\theta) := f(a, \theta x), \quad \text{with} \quad \frac{d\kappa}{d\theta} = f_x(a, \theta x) x.$$

Then

$$f(a, x) = \kappa(1) - \kappa(0) = \int_0^1 \frac{d\kappa}{d\theta} d\theta = \int_0^1 \frac{\partial f(a, \theta x)}{\partial x} x d\theta,$$

proving (3.4).

If $f \in C^k$ with $k \geq 1$, then $g \in C^{k-1}$. The set $\{f = 0\}$ is the union of $\{x = 0\}$ and the set $\{g = 0\}$.

Differentiating with respect to x yields $f_x = g + xg_x$. Therefore $f_x(\underline{a}, 0) = g(\underline{a}, 0)$. *The necessary condition $f_x(\underline{a}, \underline{x}) = 0$ for an X-bifurcation is equivalent to $g(\underline{a}, \underline{x}) = 0$.*

3.1 The second branch

To investigate $\{g = 0\}$ using the Implicit Function Theorem, need to compute the partial derivatives of g . Suppose that $f \in C^k$ that is its derivatives up to order k are continuous with $k \geq 3$. Differentiating the identity $f = xg$ yields

$$f_x = g + xg_x, \quad f_a = xg_a, \quad f_{aa} = xg_{aa}, \quad f_{ax} = xg_{ax} + g_a, \quad f_{xx} = 2g_x + xg_{xx}.$$

Evaluating at $(\underline{a}, 0)$ using (3.1) yields the values

$$g, \quad 0, \quad 0, \quad g_a, \quad 2g_x.$$

Therefore

$$g_a(\underline{a}, 0) = f_{ax}(\underline{a}, 0), \quad g_x(\underline{a}, 0) = f_{xx}(\underline{a}, 0)/2. \quad (3.5)$$

The Implicit Function Theorem implies the set $\{g = 0\}$ near $(\underline{a}, 0)$ is a C^{k-1} curve $a = h(x)$ provided that $g_a(\underline{a}, \underline{x}) \neq 0$. For the original problem this yields the following result.

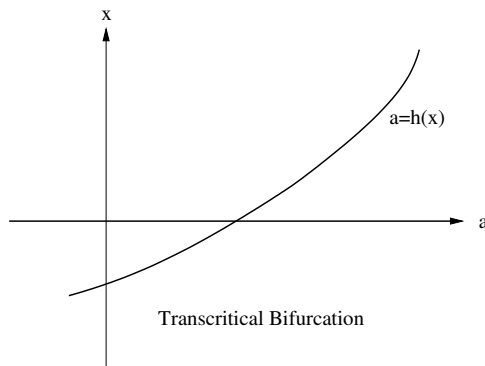
Theorem 3.1 *If in addition to (3.1) and (3.3) one has $f_{ax}(\underline{a}, 0) \neq 0$ then near $(\underline{a}, 0)$ the zero set of f consists of $\{x = 0\}$ and a C^{k-1} curve $a = h(x)$ intersecting $x = 0$ transversally at $(\underline{a}, 0)$. Precisely, $h(0) = \underline{a}$ and $h'(0) = -f_{xx}(\underline{a}, 0)/2f_{ax}(\underline{a}, 0)$.*

Exercise 3.1 . *Derive the formula for $h'(0)$. Hint.* Differentiate $g(h(x), x) = 0$ with respect to x then use (3.5).

3.2 The transcritical case

The name derives from the fact that $\{g = 0\}$ crosses the line $a = \underline{a}$ transversely with nonzero and finite slope. There are subbranches on each side of $\{a = \underline{a}\}$. The analysis is a series of exercises.

Exercise 3.2 Suppose that $f_{ax}(\underline{a}, 0) > 0$ and $f_{xx}(\underline{a}, \underline{x}) < 0$. The curve $g = 0$ then has positive slope so near $(\underline{a}, 0)$ the signs of x and $a - \underline{a}$ agree as in the figure.



Show that the equilibrium $x = 0$ is asymptotically stable for a on the left of and near \underline{a} . Show that it is unstable for $a \approx \underline{a}$ to the right. **Hint.** The stability is determined by considering the linearized equation $y' = f_x(h(x), x)y$. One needs the sign of $f_x(a, 0)$. Expand about $a = \underline{a}$. **Discussion.** This equilibrium loses its stability as a passes through \underline{a} from left to right. You should think of $f_x(a, 0)$ as a 1×1 matrix with a negative eigenvalue for $a < \underline{a}$ that crosses to positive when a increases past \underline{a} .

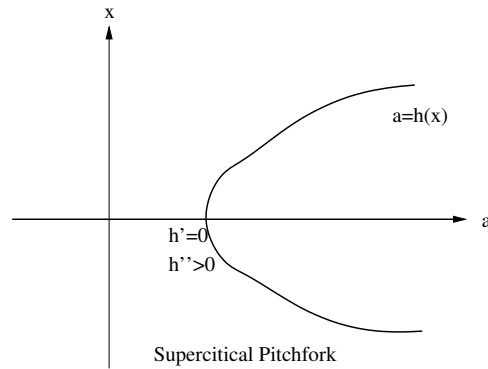
Exercise 3.3 With the hypotheses of the preceding exercise, show that the equilibria on $\{g = 0\}$ are asymptotically stable for x small positive and unstable for x small negative. Draw sketches of the branches indicating with an s or u stable and unstable branches. **Hint.** Expand $f_x(h(x), x)$ about $x = 0$. **Discussion.** The stabilities of the two halves of $\{x = 0\}$ (resp. $\{g = 0\}$) are opposed on the opposite sides of the equilibrium. The $\{x = 0\}$ branch is stable to the left and the $\{g = 0\}$ branch is stable to the right. This is called *exchange of stability*. There is an analogous exchange result when $\{x = 0\}$ is unstable to the left of $(\underline{a}, 0)$

Remark. One has analogous results whenever $f_{ax}(\underline{a}, \underline{x}) \neq 0$ and $f_{xx}(\underline{a}, \underline{x}) \neq 0$. This guarantees that $g = 0$ has finite nonzero slope at \underline{a} . The rest of the analysis is the same with care taken for all sign possibilities. A model is Exercise 3.6.

3.3 The pitchfork

The pitchfork occurs when $\nabla_{a,x}g(\underline{a}, 0) \neq 0$ so $\{g = 0\}$ is locally smooth, and in addition $\{g = 0\}$ has vertical slope and nonvanishing curvature at $(\underline{a}, 0)$. The vertical slope holds if and only if the normal to $\{g = 0\}$ at this point is horizontal, if and only if $g_x(\underline{a}, 0) = 0$. Using (3.5) this holds if and only if $f_{xx}(\underline{a}, 0) = 0$. Then thanks to (3.5), $\nabla_{a,x}g \neq 0$ holds if and only if $f_{ax}(\underline{a}, 0) \neq 0$. *These conditions are assumed for the remainder of this section.*

Then, $\{g = 0\}$ is given locally as a graph $a = h(x)$ with $h(0) = \underline{a}$ and $h'(0) = 0$. The sign of $h''(0)$ predicts which way $\{g = 0\}$ breaks in the generic case of nonvanishing curvature. The **supercritical** case $h'' > 0$ is sketched.



Exactly as in the derivation of (1.6),

$$h''(0) = \frac{-g_{xx}(\underline{a}, 0)}{g_a(\underline{a}, 0)}. \quad (3.6)$$

Exercise 3.4 Continue the computation at the beginning of §3.1 to find a formula for

$$\frac{d^3 f(h(x), x)}{d^3 x}.$$

Evaluate at $x = 0$ to show that $f_{xxx}(\underline{a}, 0) = 3 g_{xx}(\underline{a}, 0)$. Using (3.5) and (3.6) show that

$$h''(0) = \frac{-f_{xxx}(\underline{a}, 0)}{3 f_{ax}(\underline{a}, 0)}.$$

The next exercise examines the stability of the equilibria on $\{g = 0\}$. The stability is determined by the sign of f_x .

Exercise 3.5 i. Compute the derivatives of $f_x(h(x), x)$ with respect to x . Evaluate at $x = 0$. **ii.** Find the leading Taylor expansions of $f_x(a, 0)$ about $a = \underline{a}$ and $f_x(h(x), x)$ about $x = 0$. **Ans. i.**

$$\left. \frac{df_x(h(x), x)}{dx} \right|_{x=0} = 0, \quad \left. \frac{d^2 f_x(h(x), x)}{dx^2} \right|_{x=0} = \frac{2}{3} f_{xxx}(\underline{a}, 0).$$

ii. $f_x(a, 0) \approx f_{xa}(\underline{a}, 0)(a - \underline{a}), \quad f_x(h(x), x) \approx \frac{2}{3} f_{xxx}(\underline{a}, 0) x^2/2.$

Exercise 3.6 i. Use the preceding exercises together with formula (3.5) to show that the equilibria on $\{x = 0\}$ near and on opposite sides of $(\underline{a}, 0)$ have opposite asymptotic stabilities. **ii.** Show that the equilibria near $(\underline{a}, 0)$ on $\{g = 0\}$ have the same asymptotic stability as the equilibria $(a, 0)$ on the opposite side of $\{a = \underline{a}\}$. **Hint.** The pitchfork can face left or right. The equilibria on the handle can be stable or unstable. Thus there are four possibilities. Sketch some of the possibilities indicating with an s or u the stable and unstable branches. **Discussion.** Replacing f by $-f$ does not change the set of equilibria and corresponds to reversing the direction of time. It changes stable equilibria to unstable and *vice versa*. This remark reduces the set of possibilities from four to two.

4 The case $N > 1$

Surprisingly, a large part of the multidimensional case can be reduced to the scalar analysis. Consider a system of N nonlinear equations

$$X' = F(a, X), \quad F(\underline{a}, \underline{X}) = 0.$$

Here a is a real parameter and X takes values in \mathbb{R}^N .

Equilibria satisfy

$$F(a, X) = 0. \tag{4.1}$$

This is an N -vector equation that is equivalent to N scalar equations. There are N equations for the $N + 1$ unknowns a, X . The solutions set is usually a one dimensional object, a curve in $(N + 1)$ -space. The Implicit Function Theorem is the result that translates these equation count arguments into solid criteria.

The first two sections below are dedicated to the study of a single curve of equilibria. The final section describes Hopf-Andreev bifurcation which is a phenomenon not present in the scalar case $N = 1$.

4.1 The no bifurcation case

Continuing the equation count started above, for each fixed a equation (4.1) is N equations for the N unknowns. In unexceptional cases one expects to have unique solutions $K(a) \approx \underline{X}$ for $a \approx \underline{a}$. If the resulting curve of solutions $(a, X(a))$ were differentiable, then differentiating $F(a, X(a)) = 0$ with respect to a using the chain rule yields

$$F_a + F_X \frac{dK}{da} = 0, \quad \frac{dK}{da} = -(F_X)^{-1} F_a,$$

involving the $N \times N$ matrix valued function $F_X(a, X(a))$. If the $N \times N$ matrix $F_X(\underline{a}, \underline{X})$ is invertible, then the Implicit Function Theorem asserts that the set $\{F = 0\}$ of equilibria is given in a neighborhood of $(\underline{a}, \underline{X})$ by such a smooth curve

$$X = K(a), \quad \frac{dK}{da} = -(F_X(a, K(a)))^{-1} F_a(a, K(a)),$$

This is the vector version of the scalar no bifurcation result. *The criterion guaranteeing that there is no bifurcation is the invertibility of F_X .*

4.2 One curve of sub critical or super critical solutions

There is a surprising reduction to the scalar case. When F_X is not invertible, the zero set can still be a smooth curve of equilibria. The implicit function theorem assures that this is so when

$$\text{rank } F_{a,X}(\underline{a}, \underline{X}) = N. \tag{4.2}$$

On the other hand one always has

$$\text{rank } F_X(\underline{a}, \underline{X}) \leq \text{rank } F_{a,X}(\underline{a}, \underline{X}) \leq \text{rank } F_X(\underline{a}, \underline{X}) + 1$$

When F_X is not invertible it has $\text{rank} < N$ so for (4.2) to hold one must have

$$\text{rank } F_X(\underline{a}, \underline{X}) = N - 1 \tag{4.3}$$

Hypothesis (4.2)-(4.3) is equivalent to the assumption that there is a subset of $N - 1$ coordinates

$$\hat{X} := x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_N$$

and a set of $N - 1$ equations

$$\hat{F} := F_1, \dots, F_{m-1}, F_{m+1}, \dots, F_N$$

so that the $(N - 1) \times (N - 1)$ matrix $\hat{F}_{\hat{X}}(\underline{a}, \underline{X})$ is invertible. In that case the Implicit Function Theorem asserts that the set of points $\hat{F}(a, X) = 0$ is locally two dimensional parametrized by $(a, x_j) \approx (\underline{a}, \underline{x}_j)$,

$$\hat{X} = H(a, x_j) := \left(h_1(a, x_j), \dots, h_{j-1}(a, x_j), h_{j+1}(a, x_j), \dots, h_N(a, x_j) \right).$$

The equation $F(a, X) = 0$ is equivalent to $\hat{F}(a, \hat{X}) = 0$ together with the additional equation $F_m(a, X) = 0$. Define

$$\gamma(a, x_j) := F_m \left(a, h_1(a, x_j), \dots, h_{m-1}(a, x_j), x_j, h_{m+1}(a, x_j), \dots, h_N(a, x_j) \right).$$

Then $F(a, X) = 0$ is equivalent to

$$\gamma(a, x_j) = 0. \tag{4.4}$$

Equation (4.4) is a scalar equation in two variables that is analysed exactly as in §3. When the solution set is of the form $a = h(x_j)$ with $h'(x_j) = 0$ and $h''(x_j) > 0$ one finds supercritical bifurcation. The subcritical case is $h''(x_j) < 0$.

The analysis of stability of the equilibria is somewhat more subtle. The matrix of the linearization is $F_X(a, X)$. By hypothesis, 0 is an eigenvalue of multiplicity one at $\underline{a}, \underline{X}$. Using perturbation theory of eigenvalues one can show that a real eigenvalue crosses the imaginary axis as a increases through \underline{a} . If the equilibrium is stable for $a < \underline{a}$, $a \approx \underline{a}$ one concludes instability for a just to the right of \underline{a} . In general the stable manifold has dimension that increases or decreases by one when a passes through \underline{a} . The interested reader is referred to texts on Bifurcation Theory for this perturbation theory computation.

4.3 Transcritical and pitchfork bifurcations

Continuing the reduction from the last subsection, one can consider the case where $X = 0$ is an equilibrium for all a . Then one must have $H(a, 0) = 0$ and $\gamma(a, 0) = 0$ for all a . One can then factor $\gamma(a, x_j) = x_j g(a, x_j)$ and any second branch of equilibria is defined by $g(a, x_j) = 0$. This is analyzed as in the one dimensional case to reveal transcritical and pitchfork bifurcations.

Using perturbation theory of eigenvalues one can show that the $N > 1$ analogue of exchange of stability in the one dimensional case is that F_X has a real eigenvalue that changes sign at \underline{a} .

Example 4.1 *A saddle can turn to a sink or a source. This is called a saddle-node bifurcation.*

4.4 The Hopf-Andreev Bifurcation

There is a phenomenon that can occur for $N > 1$ and not for $N = 1$. The dimension of the stable manifold is the number of eigenvalues of F_X that lie in $\{\text{Re } z < 0\}$. This can change by an

eigenvalue passing through 0. This is analogous to the dimension 1 case and is discussed in the preceding subsections. There is a second mechanism when $N > 1$. A pair of complex conjugate eigenvalues can cross the imaginary axis while F_X stays invertible. In this case no new equilibria are born but the dimension of the stable manifold changes by two. If the eigenvalues cross from left to right one expects a periodic orbit to be born as a increases. In the opposite direction a periodic orbit is born as a decreases. This is called Hopf-Andreev Bifurcation.

4.4.1 Three examples of Hopf bifurcations

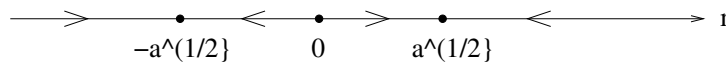
1. The youtube.com video: Vector Field: What is Hopf Bifurcation? and also HSD page 181-182 discuss the artificially simple system

$$X' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X + aX - |X|^2 X.$$

In the video, the coefficient a is called μ . The system is easily analysed since the polar coordinates satisfy

$$r' = 2r(a - r^2), \quad \theta' = 1.$$

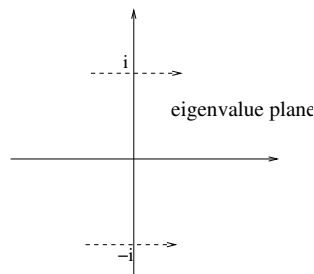
- For $a < 0$, r is strictly decreasing and all orbits converge to the origin.
- For $a > 0$ the r -phase line has a repelling equilibrium at $r = 0$ and an attracting equilibria at $r = \pm\sqrt{a}$.



Orbits starting in $0 < r < \infty$ spiral counterclockwise converging as $t \rightarrow \infty$ to the circle of radius \sqrt{a} . This is an excellent example of an attracting periodic orbit.

The system is fatally damping for $a < 0$ and becomes a self exciting oscillator for $a > 0$.

- At $a = 0$ a pair of complex conjugate eigenvalues crosses from the left half plane to the right through the eigenvalue pair $\pm i$.



The figure shows this schematically. The eigenvalues move on complex conjugate curved paths. For $0 < a \ll 1$ a periodic orbit that is the circle $r = \sqrt{a}$ emerges as a Hopf Bifurcation. The video shows the phase plane as a increases passing through 0.

2. An interesting experiment is shown in the youtube.com video entitled "Sweet Hopf Bifurcation". I think that the interpretation of the video is as follows. There is a steady motion consisting of the honey dropping vertically on a circularly symmetric pile of honey. That motion is unstable and the motion that you see is periodic in time.

If the honey were heated, the pile of honey at the bottom would become less high and the periodic motions smaller in amplitude. At a critical temperature, the bifurcation value, the amplitude of the periodic orbit would vanish and the steady circularly symmetric flow would become the stable observed motion. This is a Hopf bifurcation for an infinite dimensional system. I would like to see this more extended experiment to test my prediction.

3. A long important example is given in HSD §12.4.