Stable Manifold Difference Quotient Lemma

1 The setup

$$x' = \lambda x + f_1(x,y) := u(x,y), \qquad y' = -\mu y + f_2(x,y) := v(x,y)$$

with $-\mu < 0 < \lambda$, f_j continuously differentiable in a neighborhood of $(0,0)$ and $\nabla_{x,y} f_j(0,0) = 0$. Work in $C_M^+ \cap B_\varepsilon$ with

$$C_M^+ \ := \ \Big\{ |x| < My \ , \ y > 0 \Big\}, \qquad B_\varepsilon \ := \ \Big\{ |x_1| < \varepsilon \ , \ |x_2| < \varepsilon \Big\}.$$

Definition 1.1 *For a function h defined on $C_M^+ \cap B_\varepsilon$ we write $h = O(1)$ when there is a neighborhood of $(0,0)$ on which h is bounded. We write $h = o(1)$ when*

$$\lim_{\delta \to 0} \ \sup \Big\{ |h(x,y)| \, : \, (x,y) \in C_M^+ \cap B_\varepsilon, \ 0 < y < \delta \Big\} \ = \ 0\,.$$

Example 1.1

$$f_j = r\, o(1)\,, \qquad \nabla_{x,y} f_j = o(1)\,, \qquad \frac{r}{y} \ = \ O(1)\,.$$

Define

$$G(x,y) \ := \ \frac{\lambda x + f_1(x,y)}{-\mu y + f_2(x,y)} \ = \ \frac{u(x,y)}{v(x,y)}\,.$$

2 Difference quotient estimate

For $-y/M < x_1 < x_2 < y/M$ estimate the difference quotient

$$\frac{\Delta G}{\Delta x} \ = \ \frac{G(x_2,y) - G(x_1,y)}{x_2 - x_1}\,.$$

Lemma 2.1

$$\frac{\Delta G}{\Delta x} \ = \ \frac{\lambda}{-\mu y}\Big(1 \ + \ o(1)\Big)\,.$$

Proof. Write $x_2 = x_1 + \Delta x$, $\Delta u := u(x_1 + \Delta x, y) - u(x_1, y)$ and similarly $\Delta v := u(x_1 + \Delta x, y) - v(x_1, y)$. Then

$$\Delta G \ = \ \frac{u + \Delta u}{v + \Delta v} \ - \ \frac{u}{v} \ = \ \frac{(u + \Delta u)v - u(v + \Delta v)}{v(v + \Delta v)} \ = \ \frac{v \Delta u - u \Delta v}{v(v + \Delta v)}$$

where the two terms uv in the numerator cancelled. Therefore

$$\frac{\Delta G}{\Delta x} \ = \ \frac{v(\Delta u/\Delta x) - u(\Delta v/\Delta x)}{v(v + (\Delta v/\Delta x)\Delta x)}\,.$$

Since $f = r\, o(1)$ and $r/y = O(1)$, one has

$$u = \lambda x + r\, o(1) = r\, O(1)\,, \qquad v = -\mu y + r\, o(1) = -\mu y(1 + (r/y) o(1)) = -\mu y\big(1 + o(1)\big)\,,$$

$$\frac{1}{v} \;=\; \frac{1}{-\mu y}\,\frac{1}{1+o(1)} \;=\; \frac{1}{-\mu y}\,\big(1+o(1)\big)\,.$$

Since $\nabla_{x,y} f_2(0,0) = 0$ it follows that $\Delta v/\Delta x = o(1)$. Using also $\Delta x/y = O(1)$ yields

$$v + \frac{\Delta v}{\Delta x}\,\Delta x = -\mu y + o(1)\Delta x = -\mu y\Big(1 + o(1)\frac{\Delta x}{y}\Big) = -\mu y\big(1+o(1)\big)\,.$$

Similarly

$$\frac{\Delta u}{\Delta x} \;=\; \lambda \;+\; \frac{\Delta f_1}{\Delta x} \;=\; \lambda + o(1) \;=\; \lambda(1+o(1))\,.$$

Therefore

$$\frac{1}{v + (\Delta v/\Delta x)\Delta x} \;=\; \frac{1}{-\mu y}\,\frac{1}{1+o(1)} \;=\; \frac{1}{-\mu y}\,\big(1+o(1)\big)\,.$$

Write

$$\frac{\Delta G}{\Delta x} \;=\; \frac{\Delta u}{\Delta x}\,\frac{1}{v+(\Delta v)/\Delta x)\Delta x} \;-\; \frac{u(\Delta v/\Delta x)}{v(v+(\Delta v)/\Delta x)\Delta x)}\,. \tag{2.1}$$

Then the first summand is the product of two terms that have already been estimated,

$$\frac{\Delta u}{\Delta x}\,\frac{1}{v+(\Delta v)/\Delta x)\Delta x} \;=\; \lambda(1+o(1))\;\frac{1}{-\mu y}(1+o(1)) \;=\; \frac{\lambda}{-\mu y}(1+o(1))\,. \tag{2.2}$$

The second term is similarly the product of four terms,

$$u\,\frac{\Delta v}{\Delta x}\,\frac{1}{v}\,\frac{1}{v+(\Delta v)/\Delta x)\Delta x)} \;=\; rO(1)\;o(1)\;\frac{1}{-\mu y}(1+o(1))\;\frac{1}{-\mu y}(1+o(1))$$

Since r/y is bounded, this is equal to $1/(-\mu y)\,o(1)$. Combining this with (2.1) and (2.2) proves the lemma. ■