The Turing Instability

The Turing instability is elementary and surprising. It asserts that there are linear constant coefficient linear dynamics

$$X' \;=\; \widetilde{A}\,X, \qquad \text{and}, \qquad X' \;=\; \widetilde{B}\,X$$

both asymptotically stable, that is the eigenvalues of $\widetilde{A}$ and $\widetilde{B}$ lie in $\{\mathrm{Re}\, z < 0\}$, and so that combining the effects to yield $X' = (\widetilde{A}+\widetilde{B})X$ yields an unstable equilibrium.

We construct examples with $d = 2$. The key step is to construct A, B whose dynamics are centers and so that the sum dynamics is unstable. Let

$$A \;:=\; \begin{pmatrix} 0 & -\varepsilon \\ 1 & 0 \end{pmatrix}, \qquad 0 < \varepsilon < 1\,.$$

With $X = (x, y)$ the equation $X' = AX$ is,

$$x' = -\varepsilon y, \qquad y' = x\,.$$

Multiply the first equation by x and the second by εy and add to find,

$$0 \;=\; x\,x' \;+\; \varepsilon\, y\, y' \;=\; \frac{1}{2}\,\frac{d(x^2 \;+\; \varepsilon\, y^2)}{dt}\,.$$

On orbits, $x^2 \;+\; \varepsilon\, y^2$ is constant. The dynamics is a center and the trajectories are ellipses, with the long axis along the y axis.

A second center is defined by,

$$B \;:=\; \begin{pmatrix} 0 & 1 \\ -\varepsilon & 0 \end{pmatrix}, \qquad 0 < \varepsilon < 1\,.$$

On orbits,

$$x' = -y, \quad y' = \varepsilon\, x\,, \qquad \text{and}, \qquad \frac{d(\varepsilon\, x^2 \;+\; y^2)}{dt} \;=\; 0.$$

The trajectories are ellipses, with the long axis along the x axis.

The matrix

$$A + B \;=\; \begin{pmatrix} 0 & 1-\varepsilon \\ 1-\varepsilon & 0 \end{pmatrix}$$

is symmetric with eigenvalues $\pm(1-\varepsilon)$ of both signs so the sum dynamics is unstable. *Two centers can sum to an unstable.*

Define

$$\widetilde{A} \;:=\; A \;-\; \delta\, I\,, \qquad \widetilde{B} \;:=\; B - \delta\, I\,, \qquad 0 < \delta << 1\,.$$

The $\widetilde{A}$ and $\widetilde{B}$ have eigenvalues with real parts equal to $-\delta$. The eigenvalues of $\widetilde{A}+\widetilde{B}$ are equal to $-2\delta \pm (1-\varepsilon)$ so for $\delta < (1-\varepsilon)/2$, the larger is strictly positive. *Two exponentially asymptotically stables can sum to an unstable.*

Remark. If A and B are stable then $\mathrm{tr}\,(A+B) = \mathrm{tr}\, A + \mathrm{tr}\, B \leq 0$ so the sum cannot have both eigenvalues with positive real part.

The example is just as surprising viewed the other way. The sum of the two unstable dynamics $\widetilde{A} + \widetilde{B}$ and $-\widetilde{B}$ is stable. *The sum of two unstables can be stable.*

Exercise 1. *Show that there are real diagonal matrices with unstable dynamics whose sum defines a stable dynamics.* **Discussion.** This is much easier and less surprising than the Turing instability.

For the original A, B the matrix $A+B$ has an unstable manifold with exponential growth and that instabillity is stabilized by adding the neutrally stable $-B$. This is called *dispersive stabillization.* This phenomenon in the context of partial differential equations is studied in [1].

Summarizing, we have the the following principal.

Theorem 0.1 (Turing instability) *If $d > 1$ then knowing only the stability properties of the constant coefficient linear systems of ordinary differential equations $X' = AX$ and $X' = BX$ does not allow you to determine the stability of the system $X' = (A + B)X$.*

Turing encountered the sum of stables can be unstable in the context of reaction diffusion equations. He had a chemical reaction whose linearized behavior $u_t = Au$ at an equilibrium was stable. He added a stable but not scalar diffusion

$$u_t(t, x) = \text{diag}\,(\nu_1, \nu_2, \ldots, \nu_d)u_{xx} + Au, \qquad \nu_j > 0,$$

and found instability for the sum of the two stable processes. His classic paper on morphogenisis in which this plays a central role is [2].

If one has additional information about the matrices A and B then sometimes one can predict the stability of the sum dynamics. Two such situations are described in the next exercises.

Exercise 2. *Prove that if A and B have eigenvalues with strictly negative real parts then the same is true of $A + B$ if A and B commute.*

Exercise 3. i. *Show that if $A + A^*$ is a strictly negative hermitian symmetric matrix then $X' = AX$ has 0 as an asymptotically stable equilibrium.* **Hint.** Consider the time derivative of $\|X(t)\|^2$.

ii. *Show that if A_1 and A_2 are matrices satisfying the criterion* **i** *then the same is true of $A_1 + A_2$.*

iii. *Show that if Q is a complex scalar product and that Q is strictly decreasing on the orbits of $X' = A_1X$ and also $X' = A_2X$ then Q is also strictly decreasing on the orbits of $X' = (A_1 + A_2)X$.*
Discussion. Part **ii** is the special case where Q is the Euclidean scalar product.

Exercise 4. *Let*

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The dynamics of A is exponentially decreasing. B has only zero as eigenvalue so its dynamics is neither strongly growing or decreasing. One might guess that adding B to A would not change much the stability of A. However, the sum $A + B$ has 0 as an eigenvalue. Adding B to A nearly compensates in part the decay from A. Starting from this observation show how to make small changes in A and/or B to construct a simple example of the Turing instability.

References

[1] G. Métivier and J. Rauch, Dispersive stabilization, London Math. J. **42**(2010), 250-262.

[2] A. Turing, The chemical basis of morphogenesis, Phil. Trans. Roy. Soc. B **237**(1952), 37-72.