Mathematics 571 Winter 1999
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Assignment #12. Due April 15.

1 (Trefethen 28.1.) What happens when the QR (without shifts) is applied to a unitary matrix? Explain why this does not contradict Theorem 6.3-1.

2,3. (Trefethen 24.1, 25.1b.) For each of the following statements prove that it is true or give an example showing that it is false. Throughout A is an $n \times n$ complex matrix.

(a) If A is real and λ is an eigenvalue of A then $-\lambda$ is an eigenvalue of A.

(b) If A is invertible and λ is an eigenvalue of A then λ^{-1} is an eigenvalue of A^{-1}.

(c) If all the eigenvalues of A are equal to zero then $A = 0$.

(d) If A is diagonalizable and all the eigenvalues are equal then A is diagonal.

(e) If A is Hessenberg and all the principal subdiagonal elements $A_{i+1,i}$ are nonzero, then A has distinct eigenvalues. **Discussion.** If in addition $A = A^*$, this is proved in Theorem 6.2-2. This question asks if the result extends from hermitian Hessenberg, to arbitrary Hessenberg.

4. Let A be the tridiagonal symmetric $n \times n$ matrix with twos on the diagonal and with the upper and lower off diagonal elements all equal to one,

$$A_{i,j} = \begin{cases} 1 & \text{if } |i-j| = 1 \\ 2 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} .$$

For $n = 3^p$, $p = 1:4$ compute the eigenvalues using the QR algorithm without shifts until you believe that the errors are no larger than 1%. Discuss the number of iterations as a function of the size n. **Discussion.** You must write a QR code which takes advantage of the tridiagonal structure. Iterative eigenvalue finders are considered to be performing satisfactorily when the number of iterations is $O(n)$. For the present problem, the eigenvalues are equally spaced (the ideal situation) with $\lambda_{i+1} - \lambda_i \approx O(1/n)$ so the rate of convergence per iteration it of the order

$$\max_i \left| \frac{\lambda_{i+1}}{\lambda_i} \right| \approx \frac{1}{1 + 1/n} \approx 1 - \frac{1}{n} .$$

The number of iterations N expected for error ϵ then should satisfy

$$\epsilon \approx \left(1 - \frac{1}{n}\right)^N = \left(\left(1 - \frac{1}{n}\right)^n\right)^{N/n} \approx e^{-N/n}$$

which predicts $N/n \approx |\ln \epsilon|$ suggesting satisfactory performance.