Mathematics 571
Winter 1999
J. Rauch

Assignment #2. Due January 21.

1. For each of the following matrices determine whether it is

i. normal   ii. symmetric   iii. diagonalizable.

If diagonalizable, find the eigenvectors and eigenvalues.

$$a.\ \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad b.\ \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} \quad c.\ \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \quad d.\ \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}.$$

2. If u and v are nonzero $1 \times N$ matrices, that is column vectors, find the eigenvalues and eigenvectors of the matrix $I - u\,v^T$. **Discussion.** $N \times N$ matrix $u\,v^T$ are sometimes called *dyadic* and are sometimes written in the form $u \otimes v$ or a similar notation. They are exactly the matrices whose rank is equal to one, and this exercise amounts to computing the spectrum of rank one perturbations of the identity. Rank one perturbations of more general operators are often the most simple to analyse.

3. Show that the eigenvalues λ of an orthogonal matrix satisfy $|\lambda| = 1$. **Discussion.** So, the spectrum consists of complex conjugate pairs on the unit circle in the complex plane.

4. Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

a. What does Gershgorin's Theorem say about the eigenvalues of this matrix?

b. Let $\mathbf{e}_i$ denote the standard basis elements in $\mathbb{R}^4$. Calculate the Rayleigh quotients at the following five vectors,

$$\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_1 + \mathbf{e}_3 \quad \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4 \quad \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4\,.$$

Use this information to estimate λ_1 and λ_4, the smallest and largest eigenvalues of A. Use the Matlab eigenvalue function to compute very accurate approximate values for λ_1 and λ_4 and thereby to compute the percent error in the Rayleigh quotient estimates.

c. Compute exactly $\|A\|_1$, $\|A\|_\infty$, $\|A\|_E$, $tr(A)$. Use the results from part b., to compute approximate values of $\rho(A)$ and $\kappa_2(A)$.

Discussion. We will later find a simple formula for the eigenvalues of this matrix.