Mathematics 571
Winter 1999
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Assignment #3. Due January.

1. For exactly which complex numbers a, b, c is the matrix

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

a normal matrix.

2. Suppose that A is an $N \times N$ matrix, $x \neq 0$ is the exact solution of $Ax = b$, and $\tilde{x}$ is an approximate solution. Define the *residual* to be $r := b - A\tilde{x}$. Show that $Ae = r$.

Suppose in addtion that one has chosen a vector norm $\|\ \|$ and uses the subordinate matrix norm. Show that

$$\frac{\|e\|}{\|x\|} \le \kappa(A)\, \frac{\|r\|}{\|b\|}\,.$$

Discussion. This is my favorite expression involving the condition number.

3. Problem 2 shows that if A is invertible, $e = 0$ iff $r = 0$. However, when A is ill-conditioned the error may be large even thoughthe residual is small. This is illustrated by the following example of W. Kahan. Let

$$A := \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix}, \qquad b := \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix}, \qquad x := \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

Consider two different approximate solutions

$$\tilde{x}_1 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \tilde{x}_2 := \begin{pmatrix} 0.9911 \\ -0.4870 \end{pmatrix}.$$

Show that A is invertible and that $x = A^{-1}b$. Find $\|e\|_\infty$, and $\|r\|_\infty$, for each of the vectors $\tilde{x}_1$ and $\tilde{x}_1$. Find $\kappa_\infty(A)$. You may use Matlab.

4. Find the smallest noninvertible 2×2 matrix A whose trace is equal to 1. Justify your answer by explaining in what sense(s) the matrix is smallest.