Mathematics 571
Winter 1999
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Assignment #5. Due February 16.

1. By following the steps of the proof of Theorem 1.2-2, find the unitary matrices U and V giving the singular value decomposition of the matrix

$$A := \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}.$$

The singular values were computed in class. By multiplying the matrices, check that the formula $U^*AV = \text{diag}\,\{\mu_j\}$ is satisfied.

2. Find the LU decomposition of the matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

3. Find the BB^T decomposition of the matrix in problem 2.

4. Ciarlet problem 4.4-1. This is a computation by hand.

If A is a hermitian matrix it is important to be able to test to see that A is positive definite. This is easier than finding the eigenvalues and can be done by a direct method. For example if A_k denotes the $k \times k$ principal diagonal submatrices and $A = LU$ is the LU decomposition then we have shown that

$$U_{kk} = \frac{\det A_k}{\det A_{k-1}}, \qquad \text{and} \qquad \det A_k = U_{11} U_{22} \ldots U_{kk}.$$

By the criterion in Theorem 4.4-1, A is positive definite if and only if $\det A_k > 0$ for all k. The two formulas above show that A *is positive definite if and only if the diagonal elements of* U *are positive.* Thus performing the LU decomposition verifies positivity and then the Cholesky factor $B = L\,\text{diag}\,(U)^{1/2}$ is recovered with cost $\sim n^3/3$. However, one can halve the cost and gain stability by answering the next question.

5. If one performs the computations in the Cholesky factorization code on a symmetric but not strictly positive matrix A it must fail since a nonpositive matrix cannot be written BB^T with invertible B. Explain how this failure occurs and how it can be used to design a direct positivity test with cost $\sim n^3/6$.

Discussion. A positivity test can be used for lots of things. For example the interval $]a, b[$ has no eigenvalues of the symmetric real A if and only if $\left(A - \frac{a+b}{2} I\right)^2 - \frac{a+b}{4} I$ is positive definite. Thus you can test for such a gap in the spectrum at cost $\sim n^3/6$. This is much faster than finding eigenvalues.