Mathematics 571
Winter 1999
J. Rauch

Overflow and Study Problems

4.5-1 (part 3). Interpret this as showing that this rotation is the product of two reflections. What is the relation between the planes in which you reflect and the angle θ.

4.5-3. (Operation count for Householder elimination.)

Explain the role of condition number is the numerical solution of $Ax = b$. State at least one precise Theorem.

Can an upper triangular matrix with diagonal elements greater than or equal to 1 be ill conditioned?

2.2-6. (A 2×2 condition number exercise.)

2.2-7/part 1 (Condition number of inversion.)

Find the LU, LL^T and QR factorizations of the matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Let $\phi(x_1, x_2)$ be the steady state temperature in the L shaped region Ω defined by

$$\Omega = \Big\{(x_1, x_2) \ : \ 0 \le x_1 \le 3 \text{ and } 0 \le x_2 \le 1\Big\} \ \cup \ \Big\{(x_1, x_2) \ : \ 2 \le x_1 \le 3 \text{ and } 1 \le x_2 \le 2\Big\}.$$

The plate is heated on the left wall and cooled on the other five walls. Then ϕ is the solution of the Dirichlet problem

$$\Delta\,\phi = 0 \qquad \text{for} \qquad (x_1, x_2) \in]0,1[\times]0,1[\,,$$
$$\phi(x_1, 0) = 1\,, \qquad \text{when} \qquad x_1 = 0$$
$$\phi = 0 \qquad \text{at other boundary segments}\,.$$

Compute approximate solutions using the five point Laplacian with mesh size $h = 2^{-p}$, $p = 4, 5, 6$. Solve the linear system by Householder elimination.

Consider the strictly diagonally dominant matrices

$$A_1 := \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} \qquad A_2 := \begin{pmatrix} 1 & -3/4 \\ -1/12 & 1 \end{pmatrix}.$$

Let B_1 and B_2 denote the Jacobi iteration matrices. Show that $\rho(B_1) > \rho(B_2)$. **Discussion.** This example of R.S. Varga is intended to show that greater diagonal dominance, as in the case of A_1, does not necessarily result in more rapid convergence of the Jacobi method.

Use the results of the preceeding problem 1 to compute the spectral radius of the Gauss-Seidel iteration matrices for A_1 and A_2. **Hint.** This is a short answer question.

6.2-2 (An example symmetric eigenvalue computation by tridiagonal then diagonal reduction.)

Conte-DeBoor 4.8-13

Conte-DeBoor 4.8.14

Does every orthogonal matrix have an LU decomposition?

Repeat the calculations of problem 3 of assignment 12 using the shift strategy on the bottom of page 209. That strategy produces the eigenvalues starting with the smallest in absolute value. Compare the performance of the two methods.

Ciarlet 6.3-1, perhaps with a hint. The statement should say that P^{-1} has an LU decomposition. As of this instant the only thing that I can prove is that the statement given is false for 2 by 2's and the modified statement is true for 2 by 2's. **Discussion.** I do not yet see how to solve this problem!

Suppose that one solves the system $Ax = b$ using the QR factorization $A = QR$. The algorithm is then: x is the solution of $Rx = Q^*b$. Multilplying by the unitary matrix Q^* is safe and then solving the the equation with upper triangular R is simply back substitution. This looks pretty straight forward. Explain where the pitfall arises if the matrix A is ill conditioned.

If the matrix A has an LU decomposition and D is a diagonal matrix with nonvanishing diagonal elements show that DA and AD both have LU decompositions.

Is it possible for a matrix all of whose elements are between 0 and 1 to have an eigenvalues which is greater than 10? A negative eigenvalue?

Is it possible for a matrix A with $\|A\| \leq 1$ for a subordinate norm to have an eigenvalue of modulus greater than 2?

In class we showed that the $n \times n$ matrix with ones on the diagonal, twos on the upper diagonal and zeroes elsewhere had condition number which grows like 2^n. In contrast show that the matrix with twos on the diagonal and ones on the upper diagonal and zeros elsewhere has condition number bounded independent of n.

Perform two iterations of point Jacobi and point Gauss-Seidel for the problem

$$\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} x = b\,.$$

What is meant by an iterative method for for solving $Ax = b$. Explain the role of spectral radius in describing the convergence of such iterative methods. State at least one precise result.

For what class(es) of problems would iterative methods be particularly effective?

State a precise result relating the convergence rates of Jacobi and Gauss-Seidel methods for tridiagonal matrices. Be sure that you mention all necessary hypotheses.

Explain what can be gained by overrelaxation and what is meant by the optimal relaxation parameter. Buttress your description with personal experience from the Homework. State a set of circumstances where such an optimal parameter is described exactly.

If A is invertible and A_n is a sequence of matrices whose elements converge to those of A is it true that the condition number of A_n converges to the condition number of A? Is it true that A_n^{-1} converges to A?

What is a Householder matrix? State and prove some important properties of Householder matrices.

State the QR factorization theorem for an invertible matrix. Make sure that you include a uniqueness part.

State necessary and sufficient conditions on a matrix A for the existence of an LU factorization. If there is such a factorization what strategy might you follow to solve $Ax = b$.

For what class of matrices does the Rayleigh quotient give you eigenvalue information? What information does it give?

Why is it that eigenvalue and eigenvector algorithms must all be iterative?

Given examples of matrices whose distance apart goes to zero and so that the distance between the corresponding eigenvalues goes to zero but at a rate infinitely slower than the convergence of the matrices.

Could examples as in the previous problem be constructed which are in addition symmetric matrices. If no, state a precise result which implies the conclusion.

How is the QR factorization of an invertible matrix computed? Estimate the number of operations required.

Describe the QR algorithm (without shifts) for computing eigenvalues. If the algorithm is performed with preliminary Hessenberg reduction and the original matrix has eigenvalues of distinct modulus must the algorithm converge?

Explain how Householder matrices can be used to find a tridiagonal matrix which is similar by a unitary transformation to a given symmetric matrix A, that is $U^*AU = \text{Tridiag}$. What does a related strategy give for an arbitrary matrix A?

What is Given's recipe for computing the number of eigenvalues less that μ for a tridiagonal symmetric matrix A. Explain how this is used to construct a bisection algorithm to compute the j^{th} eigenvalue of a symmetric tridiagonal matrix.

What is the Cholesky factorization of an invertible symmetric matrix and how is it computed? What are its advantages over the LU factorization?