The Order of Accuracy of Quadrature Formulae for Periodic Functions

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We dedicate this paper to Feruccio Colombini on the event of his sixtieth birthday. He has been an inspiring colleague, coauthor and close friend. We wish him happy and creative years till the next milestone.

Abstract. The trapezoidal quadrature rule on a uniform grid has spectral accuracy when integrating C^∞ periodic function over a period. The same holds for quadrature formulae based on piecewise polynomial interpolations. In this paper, we prove that these quadratures applied to $W^{r,p}_{\rm per}$ periodic functions with $r > 2$ and $p \geq 1$ have error $\mathcal{O}((\Delta x)^r)$. The order is independent of p, sharp, and for $p < \infty$ is higher than predicted by best trigonometric approximation. For $p = 1$ it is higher by 1.

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1. Upper bound on the error

Denote by $W^{r,p}_{\rm per}$ the Banach space of periodic functions on $\mathbb{R}$ whose distribution derivatives up to order r belong to $L^p_{\rm per}(\mathbb{R})$. The norm is equal to the sum of the L^p norms of these derivatives over one period. Without loss of generality we take the period equal to 2π. Introduce a partition of the interval $[0, 2\pi]$ into N equal subintervals of size $\Delta x := 2\pi/N$.

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Because of the periodicity, the trapezoidal rule is,

$$\int_0^{2\pi} f(x)\,dx \approx T_N(f) := \Delta x \sum_{j=0}^{N-1} f(x_j), \quad x_j := j\Delta x, \quad j = 0, 1, \ldots, N-1. \tag{1.1}$$

The error is equal to

$$E_N(f) := T_N(f) - \int_0^{2\pi} f(x)\,dx.$$

Write f as the sum of its Fourier series,

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}, \qquad c_n := \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx.$$

Denote by $\mathcal{P}(m)$ the set of all trigonometric polynomials of degree at most m, that is functions of the form

$$\sum_{n=-m}^{m} a_n e^{inx}, \qquad a_n \in \mathbb{C}.$$

Summing finite geometric series shows that $T_N(e^{inx}) = 0$ for $0 < |n| < N$ and it follows that T_N exactly integrates trigonometric polynomials of degree $N-1$. Therefore, for any $P \in \mathcal{P}(N-1)$,

$$E_N(f) = E_N(f-P) = T_N(f-P) - \int_0^{2\pi} (f(x) - P(x))\,dx.$$

This yields the bound in terms of best trigonometric approximation

$$|E_N(f)| \le 4\pi \inf_{P\in\mathcal{P}(N-1)} \|f - P\|_{L^\infty}. \tag{1.2}$$

Spectral accuracy then follows for infinitely smooth f thanks to the rapid decay of the Fourier coefficients.

Our estimate proceeds differently. For any integer k the functions e^{inx} and $e^{i(n+kN)x}$ agree at the nodes for T_N. Therefore, $T_N(e^{inx}) = T_N(e^{i(n+kN)x})$ and thus, $T_N(e^{inx}) = 0$ for all $n \ne kN$ and $T_N(e^{inx}) = 2\pi$ for all $n = kN$. This can also be checked by summing the corresponding finite geometric series. It follows that

$$E_N(f) = 2\pi \sum_{0\ne k\in\mathbb{Z}} c_{kN} = 2\pi \sum_{0\ne k\in\mathbb{Z}} \Big((ikN)^r c_{kN}\Big)\frac{1}{(ikN)^r}\,. \tag{1.3}$$

This involves only a small fraction of the Fourier coefficients c_n with $|n| \ge N$.

Theorem 1.1. *If $f \in W^{r,1}_{\mathrm{per}}$ and $1 < r \in \mathbb{N}$, then the error of the trapezoidal quadrature rule (1.1) satisfies*

$$|E_N(f)| \leq \frac{C\left\|f^{(r)}\right\|_{L^1([0,2\pi])}}{N^r}, \qquad f^{(r)} := \frac{d^r f}{dx^r}, \qquad C := 2\sum_{k=1}^{\infty}\frac{1}{k^r}. \tag{1.4}$$

Remark 1.2. The result is interesting only for $r > 2$, since for $1 \leq r \leq 2$ the N^{-r} convergence rate can be established using standard arguments even in the case of nonperiodic f.

Remark 1.3. Analogous estimates are true for noninteger r. For $r \geq 2$ they can be obtained by interpolation between integer values.

Proof. Introduce

$$Q_N(x) := \frac{1}{N}\sum_{j=0}^{N-1}\delta(x - x_j), \qquad g_N(x) := \sum_{0\neq k\in\mathbb{Z}}\frac{1}{(ikN)^r}e^{ikNx},$$

where δ is the Dirac delta-function. The n^{th} Fourier coefficient of Q_N is equal to $T_N(e^{-inx})/(2\pi)^2$ so

$$Q_N(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}e^{ikNx}.$$

This together with $f^{(r)}(x) = \sum_{n\in\mathbb{Z}}(in)^r\, c_n\, e^{inx}$ yields

$$\sum_{0\neq k\in\mathbb{Z}}(ikN)^r c_{kN}\ e^{ikNx} = Q_N * f^{(r)}(x),$$

where $*$ denotes convolution of periodic functions. Equation (1.3) then implies

$$E_N(f) = \int_0^{2\pi} Q_N * f^{(r)}(x)\ g_N(-x)\,dx\,. \tag{1.5}$$

Since Q_N is a measure with total variation per period equal to 1, one has,

$$\left\|\sum_{0\neq k\in\mathbb{Z}}(ikN)^r c_{kN}e^{ikNx}\right\|_{L^1([0,2\pi])} = \left\|Q_N * f^{(r)}\right\|_{L^1([0,2\pi])} \leq \left\|f^{(r)}\right\|_{L^1([0,2\pi])}. \tag{1.6}$$

On the other hand,

$$\|g_N\|_{L^\infty} \leq \sum_{0\neq k\in\mathbb{Z}}\frac{1}{|kN|^r} = \frac{2}{N^r}\sum_{k=1}^{\infty}\frac{1}{k^r}, \tag{1.7}$$

and the sum is finite for $r > 1$. Combining (1.5), (1.6) and (1.7) yields (1.4). $\square$

Remark 1.4. The order of accuracy established in Theorem 1.1 for the trapezoidal rule, is also true for other quadratures based on piecewise polynomial interpolations. This is so since such quadratures applied to periodic functions can be rewritten as a convex combination of trapezoidal rules with shifted nodes (see [2]).

2. A lower bound and comparison with best trigonometric approximation

For integer $r > 1$, $W^{r,p}_{\rm per}$ consists of C^{r-1} functions so that $f^{(r-1)}$ is absolutely continuous with derivative in $L^p_{\rm per}$. For $p > 1$, $f^{(r-1)}$ belongs to the Hölder class $C^{r-1,\alpha}$ with $\alpha = 1 - 1/p$. The right hand side of (1.2) is largely predicted by the modulus of continuity of $f^{(r-1)}$ (see [1, 3]). The rate of best approximation is different for the different spaces $W^{r,p}_{\rm per}$ with $r > 1$ fixed and $1 \le p \le \infty$. In contrast, the order of convergence of the trapezoidal rule is essentially independent of p as the following example shows.

Example. Define f by the lacunary Fourier series:

$$f(x) := \sum_{n=1}^{\infty} \frac{1}{(2^n)^r} e^{i2^n x}.$$

Then $\int_0^{2\pi} f(x)\,dx = 0$. In addition, $f \in W^{r-\varepsilon,\infty}_{\rm per}$ for all $\varepsilon > 0$ and the error in the trapezoidal approximation $T_{2^N}(f)$ is exactly equal to

$$E_{2^N}(f) = 2\pi \sum_{2^n \text{ is a multiple of } 2^N} \frac{1}{(2^n)^r} = 2\pi \sum_{k=0}^{\infty} \frac{1}{(2^{N+k})^r} = \frac{1}{(2^N)^r} \frac{2^{r+1}\pi}{2^r - 1}.$$

As this is $\mathcal{O}((2^N)^{-r})$, the rate of convergence for $W^{r,\infty}_{\rm per}$ cannot be better, in the sense of a higher power of $1/N$, than that for $W^{r,1}_{\rm per}$.

3. Conclusion

The trapezoidal rule and other quadrature formulae based on piecewise polynomial interpolations have error $\mathcal{O}((\Delta x)^r)$ for functions in $W^{r,p}_{\rm per}$. The rate is independent of p and is optimal in the sense that no higher power of Δx is possible. The error is that which is predicted by approximation theory for functions in $W^{r,\infty}_{\rm per}$ and it is interesting that it remains true for the elements of $W^{r,1}_{\rm per}$, for which the best approximation by trigonometric polynomials of degree $N - 1$ is not as small as $\mathcal{O}((\Delta x)^{r-1+\varepsilon})$. For $W^{r,1}_{\rm per}$, the rate of convergence is essentially a full order more rapid than that given by best approximation as in (1.2).

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