1 Continuity of pseudodifferential operators

The continuity of pseudodifferential operators is a consequence, by an elegant argument of Hörmander, of the pseudodifferntial calculus.

Theorem 1.1 *Operators with symbols in S^0 are bounded operators on L^2.*

Lemma 1.1 *Operators with symbols in S^m with $m < -d$ are bounded operators on L^2.*

Proof. Formally the Schwartz kernel of $p(x, D)$ is given by the integral

$$K(x, y) \;=\; \int p(x, \xi)\; e^{i(x-y)\xi}\; dx\,.$$

When $m < d$ the integral is absoluely convergent, K is continuous, the formal computation is rigorous, and, $K \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.
An integration by parts shows that

$$\int e^{i(x-y)\xi}\; D_\xi^\alpha p(x, \xi)\, d\xi \;=\; \int (x-y)^\alpha e^{i(x-y)\xi}\; p(x, \xi)\, d\xi \;=\; (x-y)^\alpha K(x, y)\,.$$

Therefore $(x - y)^\alpha K \in L^\infty$ for all α, so

$$\forall N, \quad \exists C_N\,, \qquad |K(x, y)|| \;\le\; \frac{C_N}{1 + |x-y|^N}\,.$$

Choose $N > d$ to find the pointwise estimate

$$|p(x, D) f| \;\le\; \left(\frac{C_N}{1 + |x|^N}\right) * |f|\,.$$

Since $1/(1 + |x|^N) \in L^1$, Young's inequality implies Lemma 1.1. ■

Lemma 1.2 *Operators with symbols in S^m with $m < 0$ are bounded operators on L^2.*

Proof. Given Lemma 1.1, if suffices to show that if operators in $\mathrm{Op}S^a$ with $a < 0$ are bounded then operators in $\mathrm{Op}S^{a/2}$ are bounded. Supppse that $p(x, \xi) \in S^{a/2}$. The product rule shows that $p(x, D)^* p(x, D) \in \mathrm{Op}S^a$ so is bounded. This is equivalent to the boundedness of $p(x, D)$. ■

Proof of Theorem. Define

$$M \;:=\; 1 + \max_{x,\xi \in \mathbb{R}^{2d}} \|p(x, \xi)^* p(x, \xi)\|\,.$$

A direct verification shows that

$$q(x, \xi) \;:=\; \big(M - p(x, \xi)^* p(x, \xi)\;\big)^{1/2} \;\in\; S^0(\mathbb{R}^d \times \mathbb{R}^d)\,.$$

Then $q(x,D)^*q(x,D) \in \mathrm{Op}S^0$ with symbol

$$\big(M - p(x,\xi)^* p(x,\xi)\ \big)^{1/2} \big(M - p(x,\xi)^* p(x,\xi)\ \big)^{1/2} \ +\ S^{-1} \ =\ \big(M - p(x,\xi)^* p(x,\xi)\ \big)\ +\ S^{-1}\,.$$

Therefore

$$0\ \le\ \big(q(x,D)^* q(x,D) u\,,\mu\big)\ =\ \Big(\big[\big(M - p(x,D)^* p(x,D)\ \big)\ +\ \mathrm{Op}S^{-1}\big] u\,,\,u\Big)$$

Lemma 1.2 implies that the $\mathrm{Op}S^{-1}$ term is $\le C\|u\|^2$ as is the (Mu,u) term. Therefore

$$\Big(p(x,D)^* p(x,D) u\,,\ u\Big)\ \le\ C\,\|u\|^2$$

equivalent to the boundedness of $p(x,D)$. ■

Remark 1.1 This elegant argument proves without explaining.

Corollary 1.2 *For any s,m operators in $\mathrm{Op}S^m$ are continuous from $H^s(\mathbb{R}^d) \to H^{s-m}(\mathbb{R}^d)$.*

Proof. The operator $\langle D\rangle^a \in \mathrm{Op}S^a$ is an isometry from $H^\sigma \to H^{\sigma-a}$. The conclusion of the Corollary asserts that $\langle D\rangle^{-m+s}\,p(x,D)\,\langle D\rangle^{-s}$ is bounded from $L^2 \to L^2$. The product rule implies that the operator belongs to $\mathrm{Op}S^0$ so Theorem 1.1 implies Corollary 1.2. ■