This Gårding inequality has been improved to a sharp Gårding inequality, of the form

$$\text{Re}\,\big(p(x,D)u,u\big) \geq -C\|u\|_{L^2}^2 \quad \text{when Re}\ p(x,\xi) \geq 0, \tag{6.7}$$

first for scalar $p(x,\xi) \in S^1_{1,0}$, by Hörmander, then for matrix-valued symbols, with Re $p(x,\xi)$ standing for $(1/2)\big(p(x,\xi)+p(x,\xi)^*\big)$, by P. Lax and L. Nirenberg. Proofs and some implications can be found in Vol. 3 of [Ho5], and in [T1] and [Tre]. A very strong improvement due to C. Fefferman and D. Phong [FP] is that (6.7) holds for scalar $p(x,\xi) \in S^2_{1,0}$. See also [Ho5] and [F] for further discussion.

Exercises

1. Suppose $m > 0$ and $p(x,D) \in OPS^m_{1,0}$ has a symbol satisfying (6.1). Examine the solvability of
$$\frac{\partial u}{\partial t} = p(x,D)u,$$
for $u = u(t,x)$, $u(0,x) = f \in H^s(\mathbb{R}^n)$.
(*Hint*: Look ahead at §7 for some useful techniques. Solve
$$\frac{\partial u_\varepsilon}{\partial t} = J_\varepsilon p(x,D) J_\varepsilon u_\varepsilon$$
and estimate $(d/dt)\|\Lambda^s u_\epsilon(t)\|_{L^2}^2$, making use of Gårding's inequality.)

7. Hyperbolic evolution equations

In this section we examine first-order systems of the form

$$\frac{\partial u}{\partial t} = L(t,x,D_x)u + g(t,x), \quad u(0) = f. \tag{7.1}$$

We assume $L(t,x,\xi) \in S^1_{1,0}$, with smooth dependence on t, so

$$|D_t^j D_x^\beta D_\xi^\alpha L(t,x,\xi)| \leq C_{j\alpha\beta}\langle\xi\rangle^{1-|\alpha|}. \tag{7.2}$$

Here $L(t,x,\xi)$ is a $K \times K$ matrix-valued function, and we make the hypothesis of symmetric hyperbolicity:

$$L(t,x,\xi)^* + L(t,x,\xi) \in S^0_{1,0}. \tag{7.3}$$

We suppose $f \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, $g \in C(\mathbb{R}, H^s(\mathbb{R}^n))$.

Our strategy will be to obtain a solution to (7.1) as a limit of solutions u_ε to

$$\frac{\partial u_\varepsilon}{\partial t} = J_\varepsilon L J_\varepsilon u_\varepsilon + g, \quad u_\varepsilon(0) = f, \tag{7.4}$$

where

$$J_\varepsilon = \varphi(\varepsilon D_x), \tag{7.5}$$

for some $\varphi(\xi) \in \mathcal{S}(\mathbb{R}^n)$, $\varphi(0) = 1$. The family of operators J_ε is called a *Friedrichs mollifier*. Note that, for any $\varepsilon > 0$, $J_\varepsilon \in OPS^{-\infty}$, while, for $\varepsilon \in (0, 1]$, J_ε is bounded in $OPS^0_{1,0}$.

For any $\varepsilon > 0$, $J_\varepsilon L J_\varepsilon$ is a bounded linear operator on each H^s, and solvability of (7.4) is elementary. Our next task is to obtain estimates on u_ε, independent of $\varepsilon \in (0, 1]$. Use the norm $\|u\|_{H^s} = \|\Lambda^s u\|_{L^2}$. We derive an estimate for

$$\frac{d}{dt}\|\Lambda^s u_\varepsilon(t)\|^2_{L^2} = 2\,\mathrm{Re}\,(\Lambda^s J_\varepsilon L J_\varepsilon u_\varepsilon, \Lambda^s u_\varepsilon) + 2\,\mathrm{Re}\,(\Lambda^s g, \Lambda^s u_\varepsilon). \tag{7.6}$$

Write the first two terms on the right as the real part of

$$2(L\Lambda^s J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon) + 2([\Lambda^s, L] J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon). \tag{7.7}$$

By (7.3), $L + L^* = B(t, x, D) \in OPS^0_{1,0}$, so the first term in (7.7) is equal to

$$\big(B(t, x, D)\Lambda^s J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon\big) \le C\|J_\varepsilon u_\varepsilon\|^2_{H^s}. \tag{7.8}$$

Meanwhile, $[\Lambda^s, L] \in OPS^s_{1,0}$, so the second term in (7.7) is also bounded by the right side of (7.8). Applying Cauchy's inequality to $2(\Lambda^s g, \Lambda^s u_\varepsilon)$, we obtain

$$\frac{d}{dt}\|\Lambda^s u_\varepsilon(t)\|^2_{L^2} \le C\|\Lambda^s u_\varepsilon(t)\|^2_{L^2} + C\|g(t)\|^2_{H^s}. \tag{7.9}$$

Thus Gronwall's inequality yields an estimate

$$\|u_\varepsilon(t)\|^2_{H^s} \le C(t)\big[\|f\|^2_{H^s} + \|g\|^2_{C([0,t],H^s)}\big], \tag{7.10}$$

independent of $\varepsilon \in (0, 1]$. We are now prepared to establish the following existence result.

Proposition 7.1. *If (7.1) is symmetric hyperbolic and*

$$f \in H^s(\mathbb{R}^n), \quad g \in C(\mathbb{R}, H^s(\mathbb{R}^n)), \quad s \in \mathbb{R},$$

then there is a solution u to (7.1), satisfying

$$u \in L^\infty_{loc}(\mathbb{R}, H^s(\mathbb{R}^n)) \cap \mathit{Lip}\,(\mathbb{R}, H^{s-1}(\mathbb{R}^n)). \tag{7.11}$$

Proof. Take $I = [-T, T]$. The bounded family

$$u_\varepsilon \in C(I, H^s) \cap C^1(I, H^{s-1})$$

will have a weak limit point u satisfying (7.11), and it is easy to verify that such u solves (7.1). As for the bound on $[-T, 0]$, this follows from the invariance of the class of hyperbolic equations under time reversal.

Analogous energy estimates can establish the uniqueness of such a solution u and rates of convergence of $u_\varepsilon \to u$ as $\varepsilon \to 0$. Also, (7.11) can be improved to

$$u \in C(\mathbb{R}, H^s(\mathbb{R}^n)) \cap C^1(\mathbb{R}, H^{s-1}(\mathbb{R}^n)). \tag{7.12}$$

To see this, let $f_j \in H^{s+1}$, $f_j \to f$ in H^s, and let u_j solve (7.1) with $u_j(0) = f_j$. Then each u_j belongs to $L^\infty_{\text{loc}}(\mathbb{R}, H^{s+1}) \cap \text{Lip}(\mathbb{R}, H^s)$, so in particular each $u_j \in C(\mathbb{R}, H^s)$. Now $v_j = u - u_j$ solves (7.1) with $v_j(0) = f - f_j$, and $\|f - f_j\|_{H^s} \to 0$ as $j \to \infty$, so estimates arising in the proof of Proposition 7.1 imply that $\|v_j(t)\|_{H^s} \to 0$ locally uniformly in t, giving $u \in C(\mathbb{R}, H^s)$.

There are other notions of hyperbolicity. In particular, (7.1) is said to be *symmetrizable hyperbolic* if there is a $K \times K$ matrix-valued $S(t, x, \xi) \in S^0_{1,0}$ that is positive-definite and such that $S(t, x, \xi)L(t, x, \xi) = \tilde{L}(t, x, \xi)$ satisfies (7.3). Proposition 7.1 extends to the case of symmetrizable hyperbolic systems. Again, one obtains u as a limit of solutions u_ϵ to (7.4). There is one extra ingredient in the energy estimates. In this case, construct $S(t) \in OPS^0_{1,0}$, positive-definite, with symbol equal to $S(t, x, \xi)$ mod $S^{-1}_{1,0}$. For the energy estimates, replace the left side of (7.6) by

$$\frac{d}{dt}\big(\Lambda^s u_\epsilon(t), S(t)\Lambda^s u_\epsilon(t)\big)_{L^2}, \tag{7.13}$$

which can be estimated in a fashion similar to (7.7)–(7.9).

A $K \times K$ system of the form (7.1) with $L(t, x, \xi) \in S^1_{cl}$ is said to be *strictly hyperbolic* if its principal symbol $L_1(t, x, \xi)$, homogeneous of degree 1 in ξ, has K distinct, purely imaginary eigenvalues, for each x and each $\xi \neq 0$. The results above apply in this case, in view of:

Proposition 7.2. *Whenever (7.1) is strictly hyperbolic, it is symmetrizable.*

Proof. If we denote the eigenvalues of $L_1(t, x, \xi)$ by $i\lambda_\nu(t, x, \xi)$, ordered so that $\lambda_1(t, x, \xi) < \cdots < \lambda_K(t, x, \xi)$, then λ_ν are well-defined C^∞-functions of (t, x, ξ), homogeneous of degree 1 in ξ. If $P_\nu(t, x, \xi)$ are the projections onto the $i\lambda_\nu$-eigenspaces of L_1,

$$P_\nu(t, x, \xi) = \frac{1}{2\pi i}\int_{\gamma_\nu} \big(\zeta - L_1(t, x, \xi)\big)^{-1}\, d\zeta, \tag{7.14}$$

where γ_ν is a small circle about $i\lambda_\nu(t,x,\xi)$, then P_ν is smooth and homogeneous of degree 0 in ξ. Then

$$S(t,x,\xi) = \sum_j P_j(t,x,\xi)^* P_j(t,x,\xi) \tag{7.15}$$

gives the desired symmetrizer.

Higher-order, strictly hyperbolic PDE can be reduced to strictly hyperbolic, first-order systems of this nature. Thus one has an analysis of solutions to such higher-order hyperbolic equations.

Exercises

1. Carry out the reduction of a strictly hyperbolic PDE of order m to a first-order system of the form (7.1). Starting with
$$Lu = \frac{\partial^m u}{\partial y^m} + \sum_{j=0}^{m-1} A_j(y,x,D_x)\frac{\partial^j u}{\partial y^j},$$
where $A_j(y,x,D)$ has order $\le m-j$, form $v = (v_1,\dots,v_m)^t$ with
$$v_1 = \Lambda^{m-1}u,\dots,v_j = \partial_y^{j-1}\Lambda^{m-j}u,\dots,v_m = \partial_y^{m-1}u,$$
to pass from $Lu = f$ to
$$\frac{\partial v}{\partial y} = K(y,x,D_x)v + F,$$
with $F = (0,\dots,0,f)^t$. Give an appropriate definition of strict hyperbolicity in this context, and show that this first-order system is strictly hyperbolic provided L is.
2. Fix $r > 0$. Let $\gamma_r \in \mathcal{E}'(\mathbb{R}^2)$ denote the unit mass density on the circle of radius r:
$$\langle u, \gamma_r\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(r\cos\theta, r\sin\theta)\,d\theta.$$
Let $\Gamma_r u = \gamma_r * u$. Show that there exist $A_r(\xi) \in S^{-1/2}(\mathbb{R}^2)$ and $B_r(\xi) \in S^{1/2}(\mathbb{R}^2)$, such that
$$\Gamma_r = A_r(D)\cos r\sqrt{-\Delta} + B_r(D)\,\frac{\sin r\sqrt{-\Delta}}{\sqrt{-\Delta}}. \tag{7.16}$$
(*Hint*: See Exercise 1 in §7 of Chap. 6.)

8. Egorov's theorem

We want to examine the behavior of operators obtained by conjugating a pseudodifferential operator $P_0 \in OPS^m_{1,0}$ by the solution operator to a scalar hyperbolic equation of the form

$$\frac{\partial u}{\partial t} = iA(t,x,D_x)u, \tag{8.1}$$

where we assume $A = A_1 + A_0$ with

$$A_1(t,x,\xi) \in S^1_{cl} \text{ real}, \quad A_0(t,x,\xi) \in S^0_{cl}. \tag{8.2}$$

We suppose $A_1(t,x,\xi)$ is homogeneous in ξ, for $|\xi| \geq 1$. Denote by $S(t,s)$ the solution operator to (8.1), taking $u(s)$ to $u(t)$. This is a bounded operator on each Sobolev space H^σ, with inverse $S(s,t)$. Set

$$P(t) = S(t,0)P_0 S(0,t). \tag{8.3}$$

We aim to prove the following result of Y. Egorov.

Theorem 8.1. *If $P_0 = p_0(x,D) \in OPS^m_{1,0}$, then for each t, $P(t) \in OPS^m_{1,0}$, modulo a smoothing operator. The principal symbol of $P(t)$ (mod $S^{m-1}_{1,0}$) at a point (x_0,ξ_0) is equal to $p_0(y_0,\eta_0)$, where (y_0,η_0) is obtained from (x_0,ξ_0) by following the flow $\mathcal{C}(t)$ generated by the (time-dependent) Hamiltonian vector field*

$$H_{A_1(t,x,\xi)} = \sum_{j=1}^n \left(\frac{\partial A_1}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial A_1}{\partial x_j} \frac{\partial}{\partial \xi_j} \right). \tag{8.4}$$

To start the proof, differentiating (8.3) with respect to t yields

$$P'(t) = i[A(t,x,D), P(t)], \quad P(0) = P_0. \tag{8.5}$$

We will construct an approximate solution $Q(t)$ to (8.5) and then show that $Q(t) - P(t)$ is a smoothing operator.

So we are looking for $Q(t) = q(t,x,D) \in OPS^m_{1,0}$, solving

$$Q'(t) = i[A(t,x,D), Q(t)] + R(t), \quad Q(0) = P_0, \tag{8.6}$$

where $R(t)$ is a smooth family of operators in $OPS^{-\infty}$. We do this by constructing the symbol $q(t,x,\xi)$ in the form

$$q(t,x,\xi) \sim q_0(t,x,\xi) + q_1(t,x,\xi) + \cdots. \tag{8.7}$$

Now the symbol of $i[A, Q(t)]$ is of the form

$$H_{A_1} q + \{A_0, q\} + i \sum_{|\alpha| \geq 2} \frac{i^{|\alpha|}}{\alpha!} \Big(A^{(\alpha)} q_{(\alpha)} - q^{(\alpha)} A_{(\alpha)} \Big), \tag{8.8}$$

where $A^{(\alpha)} = D^\alpha_\xi A$, $A_{(\alpha)} = D^\alpha_x A$, and so on. Since we want the difference between this and $\partial q/\partial t$ to have order $-\infty$, this suggests defining $q_0(t,x,\xi)$ by

$$\Big(\frac{\partial}{\partial t} - H_{A_1} \Big) q_0(t,x,\xi) = 0, \quad q_0(0,x,\xi) = p_0(x,\xi). \tag{8.9}$$

Thus $q_0(t, x_0, \xi_0) = p_0(y_0, \eta_0)$, as in the statement of the theorem; we have $q_0(t, x, \xi) \in S^m_{1,0}$. Equation (8.9) is called a *transport equation*. Recursively, we obtain transport equations

$$\left(\frac{\partial}{\partial t} - H_{A_1}\right) q_j(t, x, \xi) = b_j(t, x, \xi), \quad q_j(0, x, \xi) = 0, \tag{8.10}$$

for $j \geq 1$, with solutions in $S^{m-j}_{1,0}$, leading to a solution to (8.6).

Finally, we show that $P(t) - Q(t)$ is a smoothing operator. Equivalently, we show that, for any $f \in H^\sigma(\mathbb{R}^n)$,

$$v(t) - w(t) = S(t, 0) P_0 f - Q(t) S(t, 0) f \in H^\infty(\mathbb{R}^n), \tag{8.11}$$

where $H^\infty(\mathbb{R}^n) = \cap_s H^s(\mathbb{R}^n)$. Note that

$$\frac{\partial v}{\partial t} = i A(t, x, D) v, \quad v(0) = P_0 f, \tag{8.12}$$

while use of (8.6) gives

$$\frac{\partial w}{\partial t} = i A(t, x, D) w + g, \quad w(0) = P_0 f, \tag{8.13}$$

where

$$g = R(t) S(t, 0) w \in C^\infty(\mathbb{R}, H^\infty(\mathbb{R}^n)). \tag{8.14}$$

Hence

$$\frac{\partial}{\partial t}(v - w) = i A(t, x, D)(v - w) - g, \quad v(0) - w(0) = 0. \tag{8.15}$$

Thus energy estimates for hyperbolic equations yield $v(t) - w(t) \in H^\infty$, for any $f \in H^\sigma(\mathbb{R}^n)$, completing the proof.

A check of the proof shows that

$$P_0 \in OPS^m_{cl} \Longrightarrow P(t) \in OPS^m_{cl}. \tag{8.16}$$

Also, the proof readily extends to yield the following:

Proposition 8.2. *With $A(t, x, D)$ as before,*

$$P_0 \in OPS^m_{\rho,\delta} \Longrightarrow P(t) \in OPS^m_{\rho,\delta} \tag{8.17}$$

provided

$$\rho > \frac{1}{2}, \quad \delta = 1 - \rho. \tag{8.18}$$

One needs $\delta = 1 - \rho$ to ensure that $p(\mathcal{C}(t)(x,\xi)) \in S^m_{\rho,\delta}$, and one needs $\rho > \delta$ to ensure that the transport equations generate $q_j(t,x,\xi)$ of progressively lower order.

Exercises

1. Let $\chi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism that is a linear map outside some compact set. Define $\chi^* : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by $\chi^* f(x) = f\big(\chi(x)\big)$. Show that

$$P \in OPS^m_{1,0} \Longrightarrow (\chi*)^{-1} P \chi^* \in OPS^m_{1,0}. \tag{8.19}$$

(*Hint*: Reduce to the case where χ is homotopic to a linear map through diffeomorphisms, and show that the result in that case is a special case of Theorem 8.1, where $A(t,x,D)$ is a t-dependent family of real vector fields on $\mathbb{R}^n$.)

2. Let $a \in C_0^\infty(\mathbb{R}^n)$, $\varphi \in C^\infty(\mathbb{R}^n)$ be real-valued, and $\nabla\varphi \neq 0$ on supp a. If $P \in OPS^m$, show that

$$P\big(a\, e^{i\lambda\varphi}\big) = b(x,\lambda)\, e^{i\lambda\varphi(x)}, \tag{8.20}$$

where

$$b(x,\lambda) \sim \lambda^m\big[b_0^\pm(x) + b_1^\pm(x)\lambda^{-1} + \cdots\big], \quad \lambda \to \pm\infty. \tag{8.21}$$

(*Hint*: Using a partition of unity and Exercise 1, reduce to the case $\varphi(x) = x\cdot\xi$, for some $\xi \in \mathbb{R}^n \setminus 0$.)

3. If a and φ are as in Exercise 2 above and Γ_r is as in Exercise 2 of §7, show that, mod $O(\lambda^{-\infty})$,

$$\Gamma_r\big(a\, e^{i\lambda\varphi}\big) = \cos r\sqrt{-\Delta}\big(A_r(x,\lambda)e^{i\lambda\varphi}\big) + \frac{\sin r\sqrt{-\Delta}}{\sqrt{-\Delta}}\big(B_r(x,\lambda)e^{i\lambda\varphi}\big), \tag{8.22}$$

where

$$A_r(x,\lambda) \sim \lambda^{-1/2}\big[a_{0r}^\pm(x) + a_{1r}^\pm(x)\lambda^{-1} + \cdots\big],$$
$$B_r(x,\lambda) \sim \lambda^{1/2}\big[b_{0r}^\pm(x) + b_{1r}^\pm(x)\lambda^{-1} + \cdots\big],$$

as $\lambda \to \pm\infty$.

9. Microlocal regularity

We define the notion of wave front set of a distribution $u \in H^{-\infty}(\mathbb{R}^n) = \cup_s H^s(\mathbb{R}^n)$, which refines the notion of singular support. If $p(x,\xi) \in S^m$ has principal symbol $p_m(x,\xi)$, homogeneous in ξ, then the characteristic set of $P = p(x,D)$ is given by

$$\text{Char}\, P = \{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0) : p_m(x,\xi) = 0\}. \tag{9.1}$$