1 Mike Taylor Commutator Lemma

Theorem 1.1 *Suppose that* $\mathbb{V}$ *and* $\mathbb{W}$ *are complex finite dimensional vector spaces and that* $E \in \mathrm{Hom}(\mathbb{V})$ *and* $F \in \mathrm{Hom}(\mathbb{W})$*. Then the following are equivalent.*
i. *The mapping*

$$T \ \mapsto \ \varphi(T) \ := \ FT - TE$$

is an invertible linear map from $\mathrm{Hom}(\mathbb{V}, \mathbb{W})$ *to itself.*
ii. *The spectra of* E *and* F *are disjoint.*

Proof. **ii** $\Rightarrow$ **i**. Need to show that when the spectra are disjoint, $\varphi(T) = 0$ implies $T = 0$. $\varphi(T) = 0$ yields $FT = TE$ whence

$$F^2T \ = \ F(FT) \ = \ F(TE) \ = \ (FT)E \ = \ (TE)E \ = \ TE^2 .$$

An induction proves that for $n \geq 1$, $F^nT = TE^n$. Since $I_{\mathbb{W}}T = TI_{\mathbb{V}} = T$ it follows that for any polynomial $p(z)$,

$$p(F)T \ = \ Tp(E) .$$

Choose $p(z) = \det(zI_{\mathbb{V}} - E)$ with $p(E) = 0$ to find $p(F)T = 0$.
Denote by λ_j the roots of p repeated according to their multiplicity so $p = \Pi_j(z - \lambda_j)$. The λ_j are eigenvalues of E so by hypothesis $\lambda_j I_{\mathbb{W}} - F$ is invertible for all j. Then $p(F) = \Pi_j(\lambda_j I_{\mathbb{W}} - F)$ is invertible. Mulitplying $p(F)T = 0$ on the left by $p(F)^{-1}$ yields $T = 0$ completing the proof.

i $\Rightarrow$ **ii**. Prove the contrapositive. If there is a λ in the spectrum of both E and F choose non zero $\underline{v}_1 \in \mathbb{V}$ and $\underline{w}_1 \in \mathbb{W}$ so that

$$E\underline{v}_1 \ = \ \lambda \underline{v}_1 , \qquad \text{and} \qquad F\underline{w}_1 \ = \ \lambda \underline{w}_1 .$$

Choose bases $\{\underline{v}_j\}$ and $\{\underline{w}_k\}$ starting with $\underline{v}_1$ and $\underline{w}_1$ respectively. The matrices of both E and F have λ in the upper left corner and zeros in the rest of the first column.
Choose $T \neq 0$ to be the transformation whose matrix has $T_{11} = 1$ and all other entries equal to zero. Then

$$FT \ = \ TE \ = \ \lambda T$$

so $\varphi(T) = 0$. Therefore the mapping φ is not injective. ∎