Rays, Plane Waves, and Control

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07 aprile, 2011

2. Overview

1. Characteristics and more generally **rays** very important is deciphering hyperbolic partial differential equations. We review the method of characteristics and derive sharp speed estimates in $1d$ via the Haar inequality.

2. In science texts **plane waves** play THE central role in deciphering differential equations. We illustrate by example that this is a good approach.

3. **Control** theory is introduced for linear ordinary differential equations. Duality methods appear.

4. The control of solutions of the wave equation. First in the one dimensional case using the method of characteristics. Then a result of Bardos, Lebeau, Rauch.

3. Simplest wave equation

$$\partial_t u \;+\; \partial_x u \;=\; 0, \qquad \text{general solution :} \quad u \;=\; f(x-t)\,.$$

Speed one. Arbitrary profile (good for communication).

More generally

$$\partial_t u + c(t,x)\partial_x u + d(t,x)u \;=\; 0\,, \qquad c, d \text{ real and } C^\infty \cap L^\infty\,.$$

Characteristic curves or **rays**. $(t, x(t)), \quad x' \;=\; c(t, x(t))$

Integral curves of the vector field $\partial_t + c(t,x)\partial_x$. Then

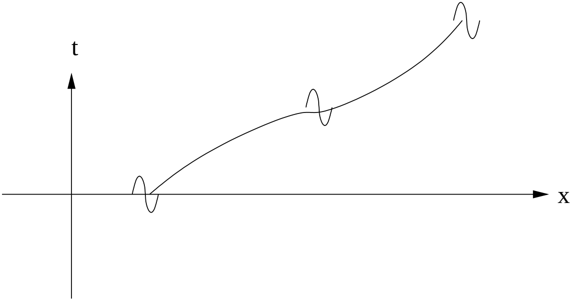
$$\frac{du(t,x(t))}{dt} \;+\; d(t,x(t))\,u(t,x(t)) \;=\; 0,$$

Linear ordinary differential equations on rays solves the initial value problem.

4. Qualitative behavior of solutions

• Waves propagate with local speed $c(t, x)$.

• Grow where $d < 0$, shrink where $d > 0$, and have amplitude preserved if $d = 0$ as in the figure.



5. Method of characteristics

$$\partial_t u \ + \ A\partial_x u \ = \ 0\,, \qquad u(t,x) \ = \ (u_1, \ldots, u_N)\,, \qquad A_{N\times N}$$

Change variable $u = Mv$. Multiply equation by M^{-1}.

$$M^{-1}\Big[\partial_t Mv + A\partial_x Mv\Big] = 0, \qquad \partial_t v + M^{-1}AM\, v = 0\,.$$

If A has N distinct real eigenvalues $c_1 < c_2 < \cdots$ choose M with

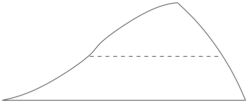
$$M^{-1}AM = \begin{pmatrix} c_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_N \end{pmatrix}, \quad \text{then} \quad (\partial_t + c_j\,\partial_x)v_j \ = \ 0\,.$$

Eigenvalues of A *are speeds.* If $A(t,x)$ has distinct real eigenvalues $c_j(t,x) < c_{j+1}(t,x)$ choose $M(t,x)$ with $M^{-1}AM$ diagonal. Then $v := Mv$ satisfies a coupled system

$$L\,v \ := \ \partial_t v \ + \ \begin{pmatrix} c_1(t,x) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_N(t,x) \end{pmatrix} \partial_x v + \widetilde{B}v \ = \ 0\,.$$

I is a closed interval in $\mathbb{R}$. Draw $\partial_t + c_N \partial_x$ ray from the left and the $\partial_t + c_1 \partial_x$ from the right. Denote by Ω the triangular domain. Ω_t its section at time t.

$$
\begin{aligned}
C &:= \sup_t \sum_j \|\widetilde{B}_{ij}(t,x)\|_{L^\infty(\Omega_T)} \\
m(t) &:= \max_{\Omega_t} \ \max_j \ |v_j(t,x)|
\end{aligned}
$$



Theorem (Haar 1928)

Solutions of $Lv = 0$ *satisfy* $m(t) \le e^{Ct}\, m(0)$.

Pf. Given t choose x and j so $|v_j(t,x)| = m(t)$. On the j ray to (t,x)

$$
\left| \frac{dv_j(t, x(t))}{dt} \right| = \left| \partial_t v_j + c_j \partial_j v_j \right| = \left| (Bv)_j \right| \le Cm(t)
$$

Integrate from $t = 0$ to t to find

$$
m(t) = |v_j(t,x)| \le m(0) + C \int_0^t m(s)\, ds. \qquad \text{Apply Gronwall.} \qquad \square
$$

7. Plane waves yield speeds in 1*d*

Translation and dilation invariant equation $\partial_t u + A\partial_x u = 0$. For $\xi \in \mathbb{R} \setminus 0$ seek plane wave solutions

$$u \;=\; e^{i(t\tau + x\xi)}\,\mathbf{u}, \quad \mathbf{u} \in \mathbb{C}^N. \quad \text{Need} \quad (\tau I + \xi A)\mathbf{u} = 0$$

Characteristic equation $0 = p(\tau, \xi) := \det(\tau I + \xi A)$. Solution set is the characteristic variety. For ξ fixed there are n roots (usually distinct) with τ equal to the negatives of the eigenvalues of ξA.

Suppose as in the method of characteristics that there are distinct real roots $\tau = -c_j \xi$ (def: strict hyperbolicity).

The corresponding solution u with $\mathbf{u}$ an eigenvector is a function of

$$t\tau + x\xi = t(-c_j\xi) + x\xi = \xi(x - c_j t)$$

It moves with speed c_j.

Plane waves yield the speeds in $1d$.

8. Plane waves and hyperbolicity

Hyperbollicity is defined by the absence of plane waves bounded in x and growing too fast in time.

Theorem (Hadamard)

If $u_t + Au_x = 0$ has a plane wave solution $e^{i(t\tau + x\xi)}\mathbf{u}$ with ξ real and τ complex then the initial value problem cannot have continuous dependence.

Proof. Take $\pm(\tau, \xi)$ so $Im\,(\tau) < 0$.

Then, $e^{i(t\tau + x\xi)/\epsilon}\mathbf{u}$ has C^k norm growing polynomially in $1/\epsilon$ at $t = 0$.

The C^0 norm at $t = 1$ grows like $e^{|Im\,\tau|/\epsilon}$. $\square$

9. Plane waves and the definition of ellipticity

With bounded open $\omega \subset\subset \Omega$ the elliptic regularity estimate is

$$\|\partial_{t,x} w\|_{L^2(\omega)} \ \leq \ C\Big(\|L\,w\|_{L^2(\Omega)} \ + \ \|w\|_{L^2(\Omega)} \Big)$$

Theorem

If $Lw = \partial_t w + A\partial_x w = 0$ has a plane wave solution $e^{i(t\tau + x\xi)}\mathbf{w}$ with τ, ξ real then the elliptic regularity estimate is not satisfied.

Proof. If there were such a solution then for any $0 < \epsilon$,

$$Lw^\epsilon \ := \ L\Big(e^{i(t\tau + x\xi)/\epsilon}\mathbf{w}\Big) \ = \ 0\,.$$

The left hand side of the regularity estimate applied to w^ϵ grows like $1/\epsilon$ as $\epsilon \to 0$. The right hand side is bounded. $\square$

End of introduction to partial differential equations.

10. Control of linear ODE

$\mathbb{V}$ is a finite dimensional complex vector space, $0 \neq \mathbf{e} \in \mathbb{V}$, and, $0 < \delta < T$.

Steer solutions of the differential equation $X' = AX$, $A \in Hom\,\mathbb{V}$.

Steer/control through a term $g(t)\,\mathbf{e}$ and $g \in C([0,\delta]\,;\,\mathbb{C})$

$$X' \ = \ A\,X + g(t)\,\mathbf{e}, \qquad X(0) = 0\,.$$

Goal is to achieve $X(T) = F$ for arbitrary $F \in \mathbb{V}$. The states F that one can reach are called **attainable**. If all of $\mathbb{V}$ is attainable, the problem is called **controllable**.

Introduce the linear map from control to final state

$$C([0,\delta]) \ \ni \ g \mapsto Kg \ := \ X(T) \ = \ \int_0^T e^{(T-s)A}\, g(s)\ \mathbf{e}\ ds \ \in \ \mathbb{V}\,.$$

Controllablity is equivalent to K being surjective.

Theorem. *The following are equivalent.*

i. *The problem is controllable.*

ii. *The only* $\mathbb{A}$ *invariant subspace containing* $\mathbf{e}$ *is* $\mathbb{V}$.

Pf. (Not **i** $\Rightarrow$ Not **ii**.) $\exists\, 0 \neq \gamma \in \mathbb{V}' \perp$ to achievable states. $\forall g$,

$$0 \;=\; \Big\langle \int_0^T e^{(T-s)A}\, g(s)\, \mathbf{e}\, ds\,,\, \gamma \Big\rangle \;=\; \int_0^\delta g(s)\, \big\langle \mathbf{e}\,,\, e^{(T-s)A^\dagger}\, \gamma \big\rangle\, ds$$

Thus $e^{-sA^\dagger} e^{TA^\dagger}\, \gamma \perp \mathbf{e}$ for $0 \le s \le \delta$. Define $\widetilde{\gamma} := e^{TA^\dagger}\gamma$. Then

$$\widetilde{W} \;:=\; Span\Big\{ e^{-sA^\dagger}\widetilde{\gamma} \;:\; s \in [0,\delta] \Big\} \;=\; Span\Big\{ (A^\dagger)^n\, \widetilde{\gamma} \;:\; n \ge 0 \Big\}$$

and $\widetilde{W}$ is $A^\dagger$ invariant and $0 \neq \widetilde{\gamma} \in \widetilde{W} \perp \mathbf{e}$. Therefore $\mathbb{W} := (\widetilde{W})^\perp$ is $\mathbb{A}$ invariant and $\mathbf{e} \in \mathbb{W} \perp \widetilde{\gamma}$. This proves Not **ii**.

(Not **ii** $\Rightarrow$ Not **i**.) Choose such a subspace $\mathbb{W} \neq \mathbb{V}$. Then for all t, $e^{tA}\mathbf{e} \in \mathbb{W}$. Then

$$X(T) \;=\; \int_0^T e^{(T-s)A}\, g(s)\, \mathbf{e}\, ds \;\in\; \mathbb{W} \;\neq\; \mathbb{V}\,. \qquad \square$$

12. Discussion.

Remark. The role of the duality and the adjoint evolution $e^{tA^\dagger}$ is a persistent theme in control.

Corollary

If you can control from $[0, T]$ then you can control from $[0, \delta]$ for any $\delta > 0$.

Proof. The criterion **ii** is independent of δ. □

Remark. This is not the case for many infinite dimensional problems.

13. A simple model hyperbolic control problem.

Control solutions of the speed one wave equation subject to the Dirichlet condition at the left hand boundary and zero initial condition,

$$u_{tt} - u_{xx} = 0 \ \ on \ \ 0 \le x \le 1, \quad u(t,0) = 0, \quad u(0,x) = u_t(0,x) = 0\,.$$

The control is $g(t)$ imposed at $x = 1$ as a Dirichlet condition

$$u_t(t,1) \ = \ g(t) \ \in \ L^2([0,\,T])\,.$$

The object is to steer the solution to arbitrary end data

$$\{u(T,x)\,,\ u_t(T,x)\} = \{F(x)\,,\ G(x)\} \in H^1([0,1]) \times L^2([0,1])\,.$$

Theorem

There is controllability if and only if $T \ge 2$.

14. Reflections.

In $0 < x < 1$ solutions of $u_{tt} - u_{xx}$ are sums of left and righward moving waves.

$u_t + u_x$ moves left, (pf. $(\partial_t - \partial_x)(u_t + u_x) = 0$), while $u_t - u_x$ moves right.

The leftward moving waves are reflected at $x = 0$ with coefficient of reflection equal to -1.

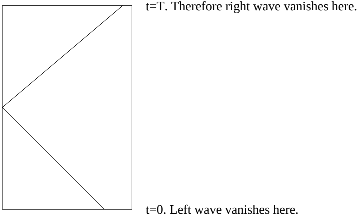
The rightward wave yields a reflected wave at $x = 1$ determined by the ougoing rightward wave and the value of g.

That is where control is exercised.

15. $(\Rightarrow)$ of controllability $\Leftrightarrow T \geq 2$.

Think backward in time.

If $2 - T = \delta > 0$ then a rightward wave supported at $t = T$ in $]1-\delta, 1[$ will have never encountered the right hand boundary in $0 \leq t \leq T$.



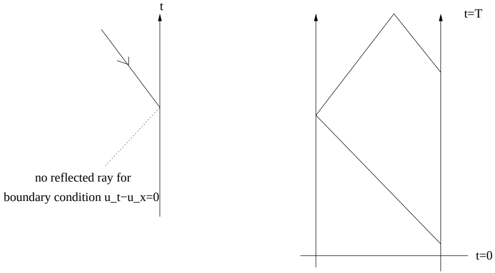
Therefore the right hand wave at $t = T$ in $]1-\delta, 1[$ is -1 times the initial leftward wave at the appropriate point.

Therefore it vanishes proving uncontrollability.

16. $(\Leftarrow)$ of controllability $\Leftrightarrow$ $T \geq 2$.

Proof of W. Littman 1978. Consider the solution v starting at time T with data F, G *and boundary condition* $v_t - v_x = 0$ *at* $x = 1$.

This says that as time decreases, if a v wave arrives at the right boundary (from the left) no reflected wave is produced. The wave is totally absorbed.



Each wave from $t = T$ is absorbed at $x = 1$, at $x = 0$, before $t = 0$ (since $T \geq 2$). Thus $v = 0$ for $t \leq 0$.

17. End proof of $\Leftarrow$.

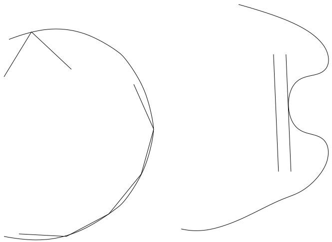
Choose the control $g(t)$ to be equal $v_t(t, 1)$. The controlled solution u is equal to v for $0 \le t \le T$.

u starts at 0 and has $\{u(T), u_t(T)\} = \{F, G\}$. Proves controllability. $\square$

Remark. Controllable with controls on $[0, T]$ but NOT for controls on $0 \le t \le \delta < 2$. Contrast ODE case.

18. $u_{tt} - \Delta u = 0$ in higher dimensions.

Rays are straight lines reflected at the boundary and their limits.
Rays carry strongly localized solutions.



Consider u defined on a nice cylindrical domain $[0, T] \times \Omega$.

Consider the boundary condition

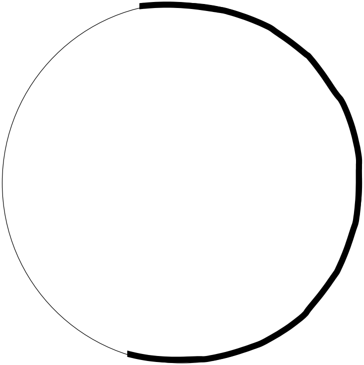
$$u_t = g \in L^2([0, T] \times \partial\Omega) \qquad supp\, g \subset [0, T] \times \Gamma$$

representing boundary control from the subset $\Gamma \subset \partial\Omega$.

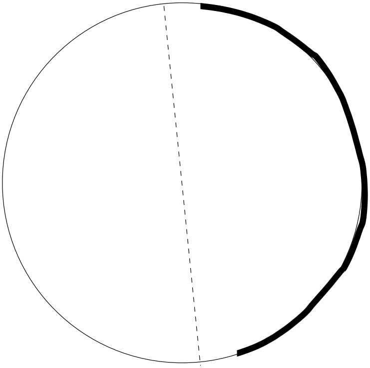
19. Geometric control.

In order to contol one must be able to influence all rays.

$[0, T] \times \Gamma$ must meet every ray.



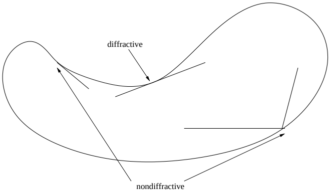
Geometric Control



No Geometric Control

Theorem (Bardos, Lebeau, R, 1992)

If $\partial\Omega$ has at most finite order contact with lines and every ray defined for $0 < t < T$ meets $]0, T[\times Int(\Gamma)$ in at least one nondiffractive point then there is controllability.



• To prove the map $Kg = \{u(T), u_t(T)\}$ is onto one proves that the transpose satisfies $\|K^\dagger w\| \;\geq\; C\|w\|, \quad C > 0\,.$

• The inequality asserts that from observations on $[0, T] \times \Gamma$ of solutions with homogeneous Dirichlet boundary condition, one can stably recover the solution. $\|\{u, u_t\}(t)\|_{H^1\times L^2} \leq C\|u_n\|_{L^2(]0,T[\times\Gamma)}$.

• The proof required new microlocal estimates.

Thanks for your attention.