C^1 Measure Respecting Maps Preserve BV Iff they have Bounded Derivative

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Abstract

If $\Omega_j \in \mathbb{R}^d$ are bounded open subsets and $\Phi \in C^1(\Omega_1\,;\,\Omega_2)$ respects Lebesgue measure and satisfies $F \circ \Phi \in BV(\Omega_1)$ for all $F \in BV(\Omega_2)$ then Φ is uniformly Lipschitzean.

The problem addressed in this note is motivated by the study of the propagation of regularity in the transport by vector fields with bounded divergence,

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{d} a_j(x,t) \frac{\partial u}{\partial x_j} \;=\; 0, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad t > 0, \tag{1}$$

where $x = (x_1, x_2, \cdots, x_d)$ and,

$$\mathbf{a} \;\in\; L^\infty\big([0,T] \times \mathbf{R}^d\big), \quad \mathrm{div}_x \mathbf{a} \;=\; \sum_{j=1}^{d} \partial_{x_j} a_j(x,t) \;\in\; L^\infty\big([0,T] \times \mathbf{R}^d\big) \tag{2}$$

The recent result of [Am] shows that this suffices to guarantee the uniqueness of L^∞ solutions of Cauchy problem.

Then, for arbitrary initial data $u_0(x) \in L^\infty(\mathbf{R}^d)$ there is a unique solution $u(x,t) \in L^\infty\big([0,T] \times \mathbf{R}^d\big)$ with $u|_{t=0} = u_0$. With the same hypotheses, there is a well defined flow Φ_t and the solution is given by $u(t) = u_0 \circ \Phi_{-t}$. The flow respects Lebesgue measure in the sense of (3) below.

We have given examples [CLR2] which show that such transport equations do not in general propagate either Hölder or BV regularity. The counterexamples had flows which were mostly smooth with small singular sets. Thus

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there were large open sets on which the flow was a C^1 maps. On those sets, the next result shows that BV preservation implies that the flow must of necessity be uniformly Lipschitzean. In the examples it is easily verified that this is violated.

The example (shown to us by L. Ambrosio) of the measure preserving map $\Phi :]0,2[\,\to\,]-1,1[$

$$\Phi(x) = x \quad \text{for} \quad 0 < x < 1\,, \quad \Phi(x) = x - 2 \quad \text{for} \quad 1 < x < 2\,,$$

shows that measure preserving maps which are smooth except for jumps, can preserve BV without being Lipschitzean.

Theorem 1. *Suppose that Ω_j are bounded open subsets of $\mathbf{R}^d$ and $\Phi \in C^1(\Omega_1\,;\,\Omega_2)$ has the following two properties*

$$\exists \gamma > 0\,, \quad \forall \text{ Borel subsets } A \subset \Omega_1, \qquad \gamma\,\big|\Phi(A)\big| \;<\; \big|A\big| \;<\; \frac{1}{\gamma}\,\big|\Phi(A)\big|\,, \tag{3}$$

where $\big|\cdot\big|$ denotes Lebesgue measure and

$$\forall F \in BV(\Omega_2), \qquad F \circ \Phi \;\in\; BV(\Omega_1)\,. \tag{4}$$

Then, $\Phi \in W^{1,\infty}(\Omega_1)$.

The proof of the Theorem consists of two lemmas.

Lemma 2. *If $\Phi \in C^1$ but not in $W^{1,\infty}$ then for any positive number M, there exists an $F \in C_0^\infty(\Omega_2)$ such that*

$$||\big(F \circ \Phi\big)'||_{L^1(\Omega_1)} \;\geq\; M\,||F'||_{L^1(\Omega_2)}. \tag{5}$$

Proof. The chain rule implies that for any $F \in C_0^1$ and $1 \leq i \leq d$,

$$\int_{\Omega_1} \Big|\frac{\partial (F\circ\Phi)(x)}{\partial x_i}\Big|\; dx = \int_{\Omega_1} \Big|\sum_{j=1}^{d} \frac{\partial F}{\partial y_j}\big(\Phi(x)\big)\; \frac{\partial \Phi_j(x)}{\partial x_i}\Big|\; dx. \tag{6}$$

Since Φ' is not bounded, there is for any $M > 0$, an $\bar{x} \in \Omega_1$ such that

$$\max_{1\leq i,\; j \leq d} \Big|\frac{\partial \Phi_i}{\partial x_j}(\bar{x})\Big| \;\geq\; \frac{8M}{\gamma}\,. \tag{7}$$

Without loss of generality, we may assume that

$$\Big|\frac{\partial \Phi_1}{\partial x_1}(\bar x)\Big| \;=\; \max_{1\le i,\ j\le d}\ \Big|\frac{\partial \Phi_i}{\partial x_j}(\bar x)\Big| \;\ge\; \frac{8M}{\gamma}\,. \tag{8}$$

Let $\bar y = (\bar y_1, \bar y_2, \cdots, \bar y_d) := \Phi(\bar x)$. For $0 < \epsilon$ small,

$$N_\epsilon \;:=\; \Big\{ y \in \mathbf{R}^d \,:\, |y_1 - \bar y_1| < \epsilon, \quad |y_j - \bar y_j| < \sqrt{\epsilon} \ \text{ for } \ 2 \le j \Big\} \;\subset \Omega_2. \tag{9}$$

Define

$$M_\epsilon \;:=\; \Phi^{-1}\Big(N_\epsilon\Big).$$

For ϵ small and $x \in M_\epsilon$,

$$\Big|\frac{\partial \Phi_1}{\partial x_1}(x)\Big| \ge \frac{1}{2}\Big|\frac{\partial \Phi_1}{\partial x_1}(\bar x)\Big|, \quad \text{and for } \ j \ge 2, \quad \Big|\frac{\partial \Phi_1}{\partial x_j}(x)\Big| \le 2\Big|\frac{\partial \Phi_1}{\partial x_1}(\bar x)\Big|. \tag{10}$$

Choose $\phi \in C_0^\infty(]-1\,,\,1[)$ satisfying

$$\int_{-\infty}^{\infty} |\phi(z)| dz \;=\; 1\,. \tag{11}$$

Define

$$F \;:=\; \phi\left(\frac{y_1 - \bar y_1}{\epsilon}\right)\ \prod_{j=2}^{d} \phi\left(\frac{y_j - \bar y_j}{\sqrt{\epsilon}}\right)\,.$$

Then,

$$\begin{aligned} ||F'\,||_{L^1(\Omega_2)} \;:=\; \int_{\Omega_2} \sum_{j=1}^{d} \ \Big|\partial_{y_j} F(y)\Big|\ dy \;&=\; \int_{N_\epsilon} \sum_{j=1}^{d}\ |\partial_{y_j} F(y)|\ dy \\ &= \epsilon^{(d-1)/2}(1 + (d-1)\sqrt{\epsilon}) \int_{-\infty}^{\infty} |\phi'(z)| dz. \end{aligned} \tag{12}$$

For ϵ small,

$$||F'\,||_{L^1(\Omega_2)} \;\le\; 2\,\epsilon^{(d-1)/2} \ \int_{-\infty}^{\infty} |\phi'(z)| dz. \tag{13}$$

In view of (**??**), (**??**) and (**??**), we have

$$\begin{aligned}
\int_{\Omega_1} \Big| \frac{\partial (F\circ\Phi)(x)}{\partial x_1}\Big| \, dx \;&=\; \int_{\Omega_1} \Big| \sum_{j=1}^{d} \frac{\partial F}{\partial y_j}\big(\Phi(x)\big)\, \frac{\partial \Phi_j(x)}{\partial x_1}\Big| \, dx \\
&\geq \int_{\Omega_1} \Big| \frac{\partial F(\Phi(x))}{\partial y_1}\, \frac{\partial \Phi_1(x)}{\partial x_1}\Big| \, dx \;-\; \int_{\Omega_1} \sum_{j=2}^{d} \Big| \frac{\partial F(\Phi(x)}{\partial y_j}\, \frac{\partial \Phi_j}{\partial x_1}\Big| \, dx \\
&= \int_{N_\epsilon} \Big| \frac{\partial F(\Phi(x))}{\partial y_1}\, \frac{\partial \Phi_1(x)}{\partial x_1}\Big| \, dx \;-\; \int_{N_\epsilon} \sum_{j=2}^{d} \Big| \frac{\partial F}{\partial y_j}\, \frac{\partial \Phi_j}{\partial x_1}\Big| \, dx \\
&\geq \Big| \frac{\partial \Phi_1(\bar{x})}{\partial x_1}\Big| \Big[\frac{1}{2} \int_{M_\epsilon} \Big| \frac{\partial F\big(\Phi(x)\big)}{\partial y_1}\Big| \, dx - 2 \int_{M_\epsilon} \sum_{j=2}^{d} \Big| \frac{\partial F\big(\Phi(x)\big)}{\partial y_j}\Big| dx \Big] .
\end{aligned}$$

Using (3) yields

$$\begin{aligned}
&\geq \Big| \frac{\partial \Phi_1(\bar{x})}{\partial x_1}\Big| \Big[\frac{\gamma}{2} \int_{N_\epsilon} \Big| \frac{\partial F(y)}{\partial y_1}\Big| \, dy - \frac{2}{\gamma} \int_{N_\epsilon} \sum_{j=2}^{d} \Big| \frac{\partial F(y)}{\partial y_j}\Big| dy \Big] \\
&= \Big| \frac{\partial \Phi_1}{\partial x_1}(\bar{x})\Big| \left(\frac{\gamma}{2} - \frac{2}{\gamma}\, \epsilon(d-1) \right) \epsilon^{(d-1)/2} \int_{-\infty}^{\infty} |\phi'(z)| dz \, .
\end{aligned}$$

Thus, for ϵ small

$$\int_{\Omega_1} \Big| \frac{\partial (F\circ\Phi)(x)}{\partial x_1}\Big| \, dx \;\geq\; \frac{\gamma}{4} \, \Big| \frac{\partial \Phi_1}{\partial x_1}(\bar{x})\Big| \, \epsilon^{(d-1)/2} \, \int_{-\infty}^{\infty} |\phi'(z)| dz. \tag{14}$$

Estimates (**??**) and (**??**) imply

$$\int_{\Omega_1} \Big| \frac{\partial (F\circ\Phi)(x)}{\partial x_1}\Big| \, dx \;\geq\; \frac{\gamma}{8} \, \Big| \frac{\partial \Phi_1}{\partial x_1}(\bar{x})\Big| \, \big\| F' \big\|_{L^1(\Omega_2)} . \tag{15}$$

(**??**) follows from (**??**) and (**??**). ∎

The next lemma completes the proof.

Lemma 3. *If $\Phi \in C^1(\Omega_1\,;\Omega_2)$ satisfies hypotheses (3) and (4) of Theorem 1, then there is a constant $C > 0$ so that for all $F \in BV(\Omega_2)$*

$$\big\| (F\circ\Phi)' \,\big\|_{\mathrm{Var}} \;\leq\; C \, \big\| F' \,\big\|_{\mathrm{Var}} \, .$$

Proof. The space of $BV(\Omega_j)$ maps H (modulo the constants) is a Banach space normed by $\|H'\|_{\mathrm{Var}}$. Thanks to the Closed Graph Theorem, it suffices to verify that the map from $BV(\Omega_2)$ to $BV(\Omega_1)$ which sends F to $F\circ\Phi$ has closed graph.

To that end, suppose that

$$F_n \quad\to\quad F \qquad \text{in} \quad BV(\Omega_2)\,,$$

and

$$F_n\circ\Phi \quad\to\quad G \quad \text{in} \quad BV(\Omega_1)\,. \tag{16}$$

It suffices to show that $G' = (F\circ\Phi)'$ in the sense of distributions.

Choose the representative $\tilde{F}_n$ of F_n and $\tilde{F}$ of F so that

$$\int_{\Omega_2} \tilde{F}_n\; dy \;=\; 0\,, \qquad \int_{\Omega_2} \tilde{F}\; dy \;=\; 0\,.$$

Then, passing to a subsequence,

$$F_{n_k} \quad\to\quad F \quad \text{in} \;\; L^1(\Omega_2)\,. \tag{17}$$

Thanks to hypothesis (3), equation (**??**) is equivalent to

$$F_{n_k}\circ\Phi \quad\to\quad F\circ\Phi \quad \text{in} \;\; L^1(\Omega_1)\,.$$

Therefore

$$(F_{n_k}\circ\Phi)' \;\to\; (F\circ\Phi)' \quad \text{in} \;\; \mathcal{D}'(\Omega_1)\,,$$

the space of distributions on Ω_1.

On the other hand (**??**) implies that,

$$(F_n\circ\Phi)' \quad\to\quad G' \quad\in\; \mathcal{D}'(\Omega_1)\,.$$

Therefore $(F\circ\Phi)' = G'$ which completes the proof. ∎

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