Hyperbolic Systems Modeling Currency Hoarding

Elena Quercioli[*], Jeffrey Rauch[†]

Dedication

It is a pleasure to dedicate this paper to Peter Lax, the greatest influence on the research of the second author. His ideas have shaped the subject of hyperbolic partial differential equations in the second part of the twentieth century. At the same time, his qualities as a person serve as a model for all scientists.

Abstract

We introduce and analyse linear systems of hyperbolic partial differential equations that model the replacement and hoarding of currency. The goal is to deduce the hoarding behavior from observations of circulating bills. The large time asymptotics of the models is identified in all cases. The mathematical analysis is novel, partly because of nonstandard boundary conditions. To identify parameters we suggest the measurement of the age histogram of notes, the rate of growth, and the retard in wear of notes due to hoarding. In our models that suffices to identify all but one quantity.

1 Introduction.

More than half of the high denomination bills printed by the US government are held out of circulation, and largely outside the US. On page v of [10],

[*]IGIER, Università Bocconi, Via Guglielmo Röntgen, 20136 Milan, Italy. email: elena.liquorice@gmail.com

[†]University of Michigan, Ann Arbor 48109 MI, USA. email: rauch@umich.edu.
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the estimate is 60%. [11] estimates that 80% of $50 bills and 90% of $100 bills are hoarded. Our goal is to extract as much information as we can from available or obtainable data.

The government acts by,

- Removing worn bills from circulation.
- Replacing with new bills. Either in the same number or with more.

People act by

- Hoarding bills which they take from circulation.
- Placing hoarded bills back in circulation.

Bills

- Wear out while in circulation and not while hoarded.

In fact, bills will deteriorate slowly when hoarded. This deterioration also applies to circulating bills. Including this slow degradation to all bills leads to models whose analysis reduces to ours.

We introduce and analyse a sequence of models of increasing complexity. From the qualitative behavior of the solutions we extract strategies for identifying parameters. All but the simplest are systems of hyperbolic partial differential equations. Two are nonstandard explicitly solvable models.

2 A model without age structure.

An ordinary differential equation model serves to introduce ideas. Denote by $c(t)$ the number of bills in circulation at time t, $h(t)$ the number hoarded, and denote,

$$Z(t) \ := \ \big(c(t), h(t)\big) \,.$$

Circulating bills are withdrawn from circulation and hoarded with rate constant a and hoarded bills reenter circulation with rate constant b. The dynamics is modelled by,

$$c' = -a\,c + b\,h\,, \qquad h' = a\,c - b\,h\,, \tag{1}$$

Equivalently,

$$Z' \;=\; M\,Z\,, \qquad \text{with,} \qquad M \;:=\; \begin{pmatrix} -a & b \\ a & -b \end{pmatrix}\,. \tag{2}$$

Proposition 1 i. *The positive quadrant $\{c \geq 0\,,\, h \geq 0\}$ is invariant, that is, a trajectory beginning in this quadrant stays in the quadrant for all future times.* **ii.** *For any solution, the quantity $c(t) + h(t)$ is independent of time.* **iii.** *As $t \to \infty$, solutions tend to the unique equilibrium with the same total number of bills.*

Proof. i. This is so since when $h = 0$ and $c > 0$, one has

$$h' \;=\; ac - bh = ac > 0\,,$$

so the trajectory through such a boundary point immediately enters the interior of the quadrant. Similarly trajectories through the boundary points $h > 0$ and $c = 0$ immediately enter the quadrant.
ii. Compute

$$(c+h)' \;=\; (-ac + bh) + (ac - bh) \;=\; 0,$$

to prove the second part.
For nonnegative solutions,

$$c(t) + h(t) \;=\; c(0) + h(0) \;=\; N\,,$$

the total number of bills.
iii. There is a unique equilibrium c_e, h_e with N bills given by solving

$$-a\,c + b\,h \;=\; 0\,, \qquad c + h \;=\; N\,,$$

to find

$$c_e := \frac{bN}{a+b}\,, \qquad h_e := \frac{aN}{a+b}\,.$$

The displacement from equilibrium

$$\delta c \;:=\; c - c_e\,, \qquad \delta h \;:=\; h - h_e\,,$$

is a vector δZ which satisfies (2). The sum of the components of δZ satisfies $\delta c + \delta h = 0$. Therefore

$$\delta c' = -a\delta c + b\delta h = -(a+b)\delta c\,.$$

so,

$$\delta c \;=\; e^{-(a+b)t}\,\delta c(0)\,, \qquad \delta h \;=\; e^{-(a+b)t}\,\delta h(0)\,.$$

Thus the displacement from equilibrium tends exponentially to zero. ∎

A surprising amount of insight can be gleaned from this model.

At equilibrium $ac = bh$ so the ratio of circulating to hoarded bills is b/a. The expected ratio of times spent circulating to time hoarded is predicted to be equal to

$$\frac{t_{\rm circ}}{t_{\rm hoard}} \;:=\; \frac{\text{time circulating}}{\text{time hoarded}} \;\approx\; \frac{b}{a}\,,$$

The chronological age is the sum

$$\text{chronological age} \;=\; t_{\rm circ} \;+\; t_{\rm hoard}\,.$$

Solving yields

$$\text{chronological age} \;=\; \Big(1+\frac{a}{b}\Big)t_{\rm circ}\,.$$

Observing the degree of wear of bills yields a measurement of $t_{\rm circ}$. *Comparing this with the chronological age of bills in circulation to yields a measurment of* b/a. This remark is independent of the model. Comparing chronological age with expected degree of wear yields a measurement of the ratio of circulating to hoarded bills.

The relatively slower decay of hoarded bills is also mirrored by the fact that low denomination bills wear much faster than large denomination bills. The average lifetimes of various denominations is (see [7]),

Denomination	1	5	10	20	50	100
Average Lifetime (months)	20	16	18.3	24.3	55.4	88.8

The absolute values of a and b are concerned with rates. Doubling a and b is equivalent to having clocks which go twice as fast. The values of a and b cannot be inferred from the equilibrium behavior. On the other hand disturbances relax to equilibrium at the rate $e^{-(a+b)t}$. It would be interesting to find a strategy for measuring this relaxation time. Similar problems reappear in the more elaborate models.

3 An age structured model without replacement.

The next model keeps track of the amount of time a bill has spent circulating in the economy. During this time, bills are subject to wear and tear. Time spent hoarded is time spent protected from wear.

Denote by $c(t,s)\,ds$ the number of bills in circulation at time t which have circulated for time in the interval $[s, s+ds]$. A typical bill in the group represented by $c(t,s)\,ds$ is older than s. It has spent s years of its life circulatiing and the rest hoarded. The number of bills at time t which have circulated for time $a \le s \le b$ is equal to

$$\int_a^b c(t,s)\,ds\,.$$

Similarly, $h(t,s)\,ds$ denotes the number of bills hoarded at time t which have circulated for time in the interval $[s, s+ds]$. The total number of bills circulating (resp. hoarded) at time t are equal to

$$C(t) := \int_0^\infty c(t,s)\ ds\,, \qquad H(t) \ := \ \int_0^\infty h(t,s)\ ds\,. \tag{3}$$

The total number of bills at time t is the sum of those circulating and hoarded so is equal to

$$\int_0^\infty c(t,s) + h(t,s)\ ds$$

We compute the equations which would be satisfied by c, h if bills circulating stay circulating and bills hoarded stay hoarded. Then $h(t,s)$ does not change with time. However the bills $c(t,s)ds$ at time $t+\Delta t$ will have been circulating for an additional Δt seconds

$$c(t+\Delta t, s+\Delta t) \ = \ c(t,s)\,. \tag{4}$$

Thus, when circulating bills and hoarded bills never change their status, the dynamics is given by

$$\partial_t c \ + \ \partial_s c \ = \ 0\,, \qquad \partial_t h \ = \ 0\,. \tag{5}$$

The first equation asserts that c is constant on lines of slope 1. If (5) is supplemented by initial data, $(c_0(s), h_0(s))$, the solution is uniquely determined by the formulas

$$c(t,s) \;=\; c_0(s-t)\,, \qquad h(t,s) \;=\; h_0(s)\,.$$

In particular if the data are nonnegative, the solution is nonnegative.

From §2, add the laws that hoarded bills enter circulation with rate b and circulating bills are hoarded with rate a. The resulting dynamics is

$$\partial_t c + \partial_s c \;=\; -a\,c + b\,h\,, \qquad \partial_t h \;=\; a\,c - b\,h\,. \tag{6}$$

Equivalently

$$\partial_t Z \;+\; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \partial_s Z \;=\; M\,Z\,. \tag{7}$$

To complete the description we suppose that no new bills are injected into circulation. This is expressed as the boundary condition

$$c(t,0) \;=\; 0\,, \quad \text{for all} \;\; t \in \mathbb{R}\,. \tag{8}$$

Proposition 2 i. *Solutions with nonnegative initial data remain nonnegative for all $t \geq 0$.* **ii.** *Equilibrium solutions depend on neither s nor t. The only equilibrium with a finite number of bills is $c = h = 0$.* **iii.** *The total number of circulating and hoarded bills, $C(t)$ and $H(t)$, satisfy the ordinary differential equation of section 1. In particular, the total number of bills is conserved and as $t \to \infty$,*

$$C(t) \;\to\; \frac{bN}{a+b}\,, \qquad \text{and} \qquad H(t) \;\to\; \frac{aN}{a+b}\,.$$

iv. *For any $A > 0$, the number of bills which have circulated for no more than A units of time is a nonincreasing function of time.*

Proof. i. This is easier than the analogous part of Proposition 9 so is omitted here.

ii. Adding the two equations (6) yields

$$\partial_t(c+h) + \partial_s c \;=\; 0 \tag{9}$$

Integrating ds shows that

$$\begin{aligned}\partial_t \int_0^\infty c(t,s) + h(t,s)\, ds \;&=\; \int_0^\infty c_t(t,s) + h_t(t,s)\, ds \\ &=\; \int_0^\infty c_s(t,x)\, ds \;=\; -\,c(t,0) \;=\; 0\,. \end{aligned} \tag{10}$$

This expresses the conservation of the total number of bills.

iii. A more general case showing that C, H satisfy the equations of section 1 is treated in section 5.3 so is omitted here.

iv. The number of bills which have circulated for no more than A years is given by

$$Y(t) := \int_0^A c(t,s) + h(t,s)\ ds.$$

Differentiating in time and using the dynamical equations yields

$$\partial_t Y \;=\; \int_0^A c_t(t,s) + h_t(t,s)\ ds \;=\; -\int_0^A \partial_s c(t,s) ds$$

The fundamental theorem of calculus shows that

$$Y_t \;=\; c(0,A) - c(t,A) \;=\; -c(t,A) \;\le 0\,,$$

proving that Y is nonincreasing. ∎

The mixed initial boudary value problem can be explicitly solved using Fourier analysis as follows. First extend $c(t,s)$ and $h(t,s)$ to $s<0$ so that

$$c(t,s) \;=\; h(t,s) \;=\; 0 \quad \text{for} \quad s \;<\; 0\,.$$

The resulting function h may be discontinuous across $s=0$ but in the dynamics, h is not differentiated in the s direction. On the other hand, the boundary condition (15) guarantees that the extended function c is continuous across $s=0$. It follows that the extended functions satisfy (6) for all $t \ge 0$ and $-\infty < s < \infty$.

Introduce the Fourier transform representations

$$Z(t,s) \;=\; \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \hat Z(t,\sigma)\ e^{is\sigma}\ d\sigma \;=\; \mathcal{F}^{-1}\,\hat Z\,,$$

$$\hat{Z}(t,s) \;=\; \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} Z(t,\sigma)\; e^{-is\sigma}\; d\sigma \;=\; \mathcal{F}\, Z\,.$$

The boundary value problem translates to the following dynamic equation for the transform,

$$\partial_t \hat{Z}(t,\sigma) \;=\; \begin{pmatrix} -a - i\sigma & b \\ a & -b \end{pmatrix} Z(t,\sigma) \;:=\; M(\sigma)\, \hat{Z}(t,\sigma)\,. \tag{11}$$

For each σ this is a linear constant coefficient system of ordinary differential equations in time which can be explicitly solved. This yields the solution formula

$$\mathcal{F}^{-1}\; \exp\big(t\, M(\sigma)\big)\; \mathcal{F}\, Z(0)\,.$$

where $\mathcal{F}$ denotes the Fourier transform. The map from initial data to solution at time t is given by a Fourier multiplier. We next analyse this to determine the behavior as $t \to \infty$.

Proposition 3 *For every real nonzero σ, the eigenvalues of $M(\sigma)$ in (11) have strictly negative real part.*

Proof. To see this start with the fact that the eigenvalues are the roots of f_σ where,

$$f_\sigma(\lambda) \;:=\; \det\big[M(\sigma) - \lambda I\big] \;=\; \big(\lambda + (a + i\sigma)\big)\big(\lambda + b\big) \;-\; ab \;:=\; g_\sigma(\lambda) - ab\,.$$

Both f_σ and g_σ are quadratic polynomials in λ. g_σ has the roots $\lambda_1 = -b$ and $\lambda_2 = -a - i\sigma$, with strictly negative real part.

On the circles $|z - \lambda_j| = \epsilon << 1$, one has for large σ, $|g_\sigma| \geq |\sigma|\epsilon$. Choosing $\epsilon = 2|ab/\sigma|$, it follows by Rouché's Theorem that f_σ has a root inside each circle. Thus, the roots are $-a - i\sigma + O(1/|\sigma|)$ and $-b + O(1/|\sigma|)$. Thus for large σ the roots lie in the left half plane.

If there were a a root in the closed right halfplane with $\underline{\sigma} > 0$ (resp. $\underline{\sigma} < 0$), then there would have to be a $\sigma \in]\underline{\sigma}, \infty[$ (resp. $\sigma \in] - \infty, \underline{\sigma}[$) for which a root crosses the imaginary axis. For this σ there is a purely imaginary root. Therefore, it suffices to show that there can be no purely imaginary root for $\sigma \neq 0$.

Inserting $\lambda = i\zeta$ with $\zeta \in \mathbb{R}$ yields the equation

$$-\zeta^2 + (a + i\sigma + b)i\zeta + (b(a + i\sigma) - ab) \;=\; 0\,.$$

Taking the real part yields

$$-\zeta^2 - \sigma\zeta = 0\,, \quad \Longrightarrow \quad \zeta = 0\,, \ \text{ or } \ \zeta = -\sigma\,. \tag{12}$$

On the other hand taking the imaginary part yields

$$(a+b)\zeta + \sigma b = 0\,, \quad \Longrightarrow \quad \zeta = \frac{-\sigma b}{a+b}\,. \tag{13}$$

Since the last conditions in (12) and (13) are mutually exclusive, there is no purely imaginary root $\lambda = i\zeta$. This completes the proof that the eigenvalues have negative real part. ∎

Proposition 4 i. *For possibly complex solutions of (6),*

$$\int_{-\infty}^{\infty} \big|\mathrm{Re}\, c(t,s)\big| + \big|\mathrm{Im}\, c(t,s)\big| + \big|\mathrm{Re}\, h(t,s)\big| + \big|\mathrm{Im}\, h(t,s)\big| \ ds$$

is a nondecreasing function of t.
ii. *For any* $1 \le p < \infty$ *and* $t \ge 0$,

$$\Big(\int_{-\infty}^{\infty} \big|\mathrm{Re}\, c(t,s)\big|^p + \big|\mathrm{Im}\, c(t,s)\big|^p + \big|\mathrm{Re}\, h(t,s)\big|^p + \big|\mathrm{Im}\, h(t,s)\big|^p\, ds\Big)^{1/p} \ \le$$
$$2\Big(\int_{-\infty}^{\infty} \big|\mathrm{Re}\, c(0,s)\big|^p + \big|\mathrm{Im}\, c(0,s)\big|^p + \big|\mathrm{Re}\, h(0,s)\big|^p + \big|\mathrm{Im}\, h(0,s)\big|^p\, ds\Big)^{1/p}.$$

iii. *For real* σ *and* $t \ge 0$,

$$\big\|\exp t\, M(\sigma)\big\| \ \le \ 2\,. \tag{14}$$

Proof. Define nonegative initial functions

$$c_1^0(s) \ := \ \max\{\mathrm{Re}\, c(0,s), 0\}\,, \qquad c_2^0(s) \ := \ -\min\{\mathrm{Re}\, c(0,s), 0\}\,,$$
$$c_3^0(s) \ := \ \max\{\mathrm{Im}\, c(0,s), 0\}\,, \qquad c_4^0(s) \ := \ -\min\{\mathrm{Im}\, c(0,s), 0\}\,,$$

with analogous $h_1^0(s)$. Define nonnegative solutions $(c_i(t,s), h_i(t,s))$ of (6) with initial data equal to $(c_i^0(s), h_i^0(s))$. Then

$$c \ = \ c_1 - c_2 + ic_3 - ic_4\,, \qquad h \ = \ h_1 - h_2 + ih_3 - ih_4\,.$$

Therefore

$$\int_{-\infty}^{\infty} \big|\mathrm{Re}\, c\big| + \big|\mathrm{Im}\, c\big| + \big|\mathrm{Re}\, h\big| + \big|\mathrm{Im}\, h\big| \ ds \ \le \ \int \sum (c_i + h_i)\, ds\,.$$

The conservation of bills implies that

$$\int_{-\infty}^{\infty} c_i(t,s) \ + \ h_i(t,s) \ ds \ = \ \int_{-\infty}^{\infty} c_i^0(s) \ + \ h_i^0(s) \ ds\,.$$

From the definition,

$$\int \sum (c_i^0 + h_i^0)\, ds = \int_{-\infty}^{\infty} \big|\mathrm{Re}\, c(0,s)\big| + \big|\mathrm{Im}\, c(0,s)\big| + \big|\mathrm{Re}\, h(0,s)\big| + \big|\mathrm{Im}\, h(0,s)\big| \ ds.$$

Combining the last three displayed equations proves **i.**

The expression

$$p(c,h) \ := \ \big|\mathrm{Re}\, c\big| + \big|\mathrm{Im}\, c\big| + \big|\mathrm{Re}\, h\big| + \big|\mathrm{Im}\, h\big|$$

defines a norm on complex two vectors (c,h). To compare with the Euclidean norm begin with

$$\begin{aligned} p(c,h)^2 \ &= \ \Big(\big|\mathrm{Re}\, c\big| + \big|\mathrm{Im}\, c\big| + \big|\mathrm{Re}\, h\big| + \big|\mathrm{Im}\, h\big| \Big)^2 \\ &\ge \ \big|\mathrm{Re}\, c\big|^2 + \big|\mathrm{Im}\, c\big|^2 + \big|\mathrm{Re}\, h\big|^2 + \big|\mathrm{Im}\, h\big|^2 \ = \ \|(c,h)\|^2\,. \end{aligned}$$

For the opposite comparison, the Cauchy-Schwarz inequality yields

$$\begin{aligned} p(c,h) \ &= \ \big|\mathrm{Re}\, c\big| + \big|\mathrm{Im}\, c\big| + \big|\mathrm{Re}\, h\big| + \big|\mathrm{Im}\, h\big| \ \le \\ &\Big(\big|\mathrm{Re}\, c\big|^2 + \big|\mathrm{Im}\, c\big|^2 + \big|\mathrm{Re}\, h\big|^2 + \big|\mathrm{Im}\, h\big|^2 \Big)^{1/2} \Big(1^2 + 1^2 + 1^2 + 1^2 \Big)^{1/2} \\ &= \ 2\,\|c,h\|\,. \end{aligned}$$

Therefore, using **i** yields,

$$\begin{aligned} \int_{-\infty}^{\infty} \|Z(t,s)\| \ ds &\le \int_{-\infty}^{\infty} p(Z(t,s))\| \ ds \\ &\le \int_{-\infty}^{\infty} p(Z(0,s))\| \ ds \le 2 \int_{-\infty}^{\infty} \|Z(0,s)\| \ ds\,. \end{aligned}$$

This is the case $p = 1$ of **ii**. A standard duality and interpolation argument shows that Fourier multipliers that satisfy the case $p = 1$ satisfy the estimate for all $p \in [0, \infty[$.

Finally, the estimate **ii** for $p = 2$ immediately implies **iii.**. ∎

Theorem 5 *Bills wear out. Precisely, if $Z(0,s)$ vanishes for $s < 0$ and $\int_{-\infty}^{\infty} |\hat{Z}(0,\sigma)|\, d\sigma < \infty$, then for any age $A > 0$,*

$$\lim_{t\to\infty} \quad \max_{0\le s\le A} |Z(t,s)| \;=\; 0\,.$$

Proof. For any challenge number $\epsilon > 0$ choose $\mu > 0$ so small that

$$\int_{|\sigma|<\mu} |\hat{Z}(0,\sigma)|\; d\sigma \;+\; \int_{|\sigma|>1/\mu} |\hat{Z}(0,\sigma)|\; d\sigma \;<\; \epsilon/4\,.$$

Part **iii** of the preceding proposition shows that for all $t > 0$ and s,

$$\Big| \int_{|\sigma|<\mu} e^{it\sigma}\ \hat{Z}(t,\sigma)|\; d\sigma \Big| + \Big| \int_{|\sigma|>1/\mu} e^{it\sigma}\ \hat{Z}(t,\sigma)|\; d\sigma \Big| \;<\; \epsilon/2\,.$$

Proposition 3 shows that as $t \to \infty$,

$$\max_{\mu<|\sigma<1/\mu} \|\exp(tM(\sigma)\| \;\to\; 0\,.$$

Therefore

$$\max_{0\le s\le A} \Big| \int_{\mu<|\sigma|<1/\mu} \exp\big(tM(\sigma)\big)\ e^{it\sigma}\ \hat{Z}(0,\sigma)\; d\sigma \Big| \;\to\; 0$$

as $t \to \infty$. This completes the proof. ∎

4 A model with bill replacement after S years of wear.

To combat aging, worn bills are removed from circulation and replaced with new ones. It is the wear on bills not their age which determines whether they are removed from circulation ([7]). The cohort of bills is rejuvenated. We study a model which like that in the preceding section is exactly solvable by Fourier analysis. It uses number of years in circulation as a measure of wear and replaces bills that have circulated for S units of time with new bills. This model has the advantage of being exactly solvable. More realistic models are discussed later.

Bills in circulation have circulated for $0 \leq s \leq S$ years so the unknowns $c(t,s)$, $h(t,s)$ are defined on this s interval. The circulating bills which arrive at circulation age S are replaced by new ones. This is expressed by the boundary condition,

$$c(t,0) \;=\; c(t,S)\,. \tag{15}$$

The model is the system of partial differential equations (6) for $0 \leq s \leq S$ *supplemented by the boundary condition (15).* The initial data $c_0(s), h_0(s)$ must satisfy (15).

There is an important subclass of solutions of this boundary value problem which are functions of t alone. The densities c, h do not depend on s. They have uniform age distribution. Each value of s is as likely as any other. Note that the boundary condition (15) is satisfied by such solutions.

In §2, we showed that such solutions tend to the unique equilibrium with the correct total number of bills. We next show that in the general case the solution tends to a uniform equilibrium age distribution with the correct total number of bills. The results of the first section then give the large time behavior.

Proposition 6 *The first three assertions of Proposition 2 hold for the solutions of the model (6) -(15).*

The proof follows exactly the lines of the earlier result and is omitted.

The boundary value problem is solved by the *method of reflection*. Extend c, h to the line $\mathbb{R}_s$ as periodic functions of period S. The resulting function is a solution of (6) on the whole s line. To reach this conclusion the boundary condition (15) is crucial. If it were not satisfied the periodic extension of $c(t,s)$ would have jump discontinuities at the points $s = nS$ and the c equation from (6) would not be satisfied at those points.

The S periodic version of (6) is solved explicitly by Fourier expansion,

$$Z(t,s) \;=\; \sum_{n=-\infty}^{\infty} Z_n(t)\, e^{i2\pi ns/S}\,. \tag{16}$$

Equation (7) holds if and only if the Fourier coefficients, $Z_n(t)$ satisfy the ordinary differential equations

$$Z_n' + \frac{2\pi ni}{S}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} Z_n \;=\; M\,Z\,, \tag{17}$$

equivalently,

$$Z_n' \;=\; \begin{pmatrix} -a - i\sigma & b \\ a & -b \end{pmatrix} Z_n \;:=\; M(\sigma)\, Z_n\,, \qquad \sigma := \frac{2\pi n i}{S}\,. \tag{18}$$

Proposition 7 *The only equilibrium solutions of the boundary value problem (6), (15) are functions which do not depend on s and are equilibria of the ordinary differential equations (1).*

Proof. This is so since an equilibrium does not depend on t so in (16) only the $n = 0$ term can occur. For this term, (18) reduces to (7). ∎

The eigenvalue information in Proposition 3 shows that for $n \neq 0$, the solutions $Z_n(t)$ of (18) tend to zero as $t \to \infty$. Therefore as $t \to \infty$

$$Z(t,s) \quad\to\quad Z_0(t)$$

a solution which is independent of s. Asymptotically the solution has uniform distribution of ages.

Writing $Z_0(t) = \big(c_0(t), h_0(t)\big)$, in §1 we showed that Z_0, which satisfies (2), tends as $t \to \infty$ to an equilibrium of (1). By the conservation of bills, it must be the unique equilibrium with the same total number of bills as the initial data. Combining these two conclusions yields the following result.

Theorem 8 *As $t \to \infty$, each solution of the boundary value problem (6), (15) tends to the unique equilibrium which has the same total number of bills as the initial data. That equilibrium has uniform age distribution.*

For this model the age histogram of bills at equilibrium is independent of s. It is as likely to find a bill which is four years old as one which is brand new. In [10] one finds the age histogram for one dollar bills in circulation,

Months	5	10	15	20	25	30	35	40	45	50	55	60	65
Number	500	1190	1600	880	1000	990	610	500	400	250	210	175	100

The nonuniformity shows that the present model is inappropriate. However, the analysis performed provides insights that serve for the better models.

5 A model with replacement for all degrees of wear.

5.1 Introducing the model.

In the preceding model, bills in circulation were removed and replaced with new bills when they have been worn for *exactly* S units of time. More realisitically, the removal rate of bills is a nondecreasing function, $\delta(s)$, of the number years that they have been wearing out. The resulting dynamics is

$$\partial_t c + \partial_s c \;=\; -a\,c + b\,h - \delta(s)\,c\,, \qquad \partial_t h \;=\; a\,c - b\,h\,. \tag{19}$$

A model which replaces the removed bills with an equal number of new ones adjoins the boundary condition

$$c(t,0) \;=\; \int_0^\infty \delta(s)\,c(t,s)\;ds\,. \tag{20}$$

In this model, $0 \leq s < \infty$ and arbitrarily old bills are permitted. The tail of the distributions as $s \to \infty$ will be shown to be exponentially small.

Initial data $c_0(s), h_0(s)$ must satisfy (20) in order to avoid discontinuous solutions.

The removal at age S model is an extreme case where the removal rate is zero for $s < S$ and completely effective for $s = S$. This is a limiting case of $\delta(s)$ models as follows. Define $H(s)$ to be the step function

$$H(s) \;:=\; \begin{cases} 0 & \text{when} \quad s < S \\ 1 & \text{when} \quad s \geq S \end{cases}.$$

In the limit $\lambda \to \infty$ the model with $\delta(s) = \lambda\,H(s)$ converges to the model of §2.

Proposition 9 i. *If c, h is a continuous solution of (19) on $[T_1, T_2[\times[0,R[$ and*

$$c\big|_{[T_1,T_2[\times\{s=0\}} \;\geq\; 0\,, \quad c\big|_{\{s=0\}\times[0,R[} \;\geq\; 0\,, \quad \text{and} \quad h\big|_{\{t=T_1\}\times[0,R[} \;\geq\; 0\,,$$

then c and h are nonnegative on the rectangle $[T_1, T_2[\times[0,R[$.

ii. *If c, h is a nonnegative continuous solution of (19) on the triangular domain of influence*

$$D \;:=\; \Big\{(t,s)\;:\;\underline{t}\le t\le \underline{T},\quad \text{and}\quad \underline{s}\le s\le s+(t-\underline{t})\Big\}\,,$$

and at least one of c or h is strictly positive at the vertex $(\underline{t},\underline{s})$, then both c and h are strictly positive at every interior point of D.

Proof. To prove **i**, consider the initial boundary value problem for the system (19) supplemented by nonnegative initial values for c and h on the base and for c on the left hand boundary. Define a first approximate solution to be $c = h = 0$ and for $\nu > 0$ define c^ν, h^ν inductively by solving

$$c^\nu_t + c^\nu_s + (a+\delta(s))c^\nu \;=\; b\,h^{\nu-1}\,,\qquad h^\nu_t + bh^\nu \;=\; a\,c^{\nu-1}\,,$$

with the given nonnegative initial and boundary values.

It is not difficult to prove that the c^ν, h^ν is nondecreasing in ν and converges to the unique solution of the boundary value problem. This proves the nonnegativity of that solution.

To prove **ii** suppose first that $c(\underline{t},\underline{s}) > 0$. Integrate

$$h_t + b\,h = a\,c \ge 0\,, \tag{21}$$

along the left boundary of D to conclude that h is strictly positive along the left boundary with the possible exception of the vertex.

Next integrate

$$c_t + c_s + (a+\delta(s))c \;=\; b\,h \ge 0 \tag{22}$$

along the speed one path connecting (t,s) to the left hand boundary. Since c is nonnegative at the boundary and source term on the right is strictly positive when the path is close to the left hand boundary, it follows that c is strictly positive at all interior points of D.

Then (21) implies that h is strictly positive in the interior of D.

When $h(\underline{t},\underline{s}) > 0$ the proof is similar. Start with the c equation to show that c is strictly positive on the nonvertex points of the right hand boundary. Then (21) shows that h is strictly positive in the interior. Then (22) shows that c is strictly positive in the interior. ∎

Proposition 10 *Suppose that c, h is a not necessarily nonnegative solution of (19), (20), which converges exponentially to zero as $s \to \infty$ uniformly in $t \geq 0$.* **i.** *For all $t \geq 0$,*

$$\int_0^\infty |c(t,s)| + |h(t,s)| \ ds \ \leq \ \int_0^\infty |c(0,s)| + |h(0,s)| \ ds \, .$$

ii. *The only case where there is equality in* **i** *for all $t \geq 0$ is when both c and h are everywhere nonnegative or everywhere nonpositive.*

Proof. **i.** Denote by $c_0^\pm(s) := \max\{\pm c(0,s), 0\}$. Similarly, $h_0^\pm(s)$. Then $c^\pm$ and $h^\pm$ are nonnegative and

$$|c(0,s)| = c_0^+(s) + c_0^-(s)\, , \qquad |h(0,s)| = h_0^+(s) + h_0^-(s)\, ,$$

so

$$\int_0^\infty |c(0,s)| + |h(0,s)| \ ds \ = \ \int_0^\infty c_0^+(s) + h_0^+(s) + c_0^-(s) + h_0^-(s) \ ds \, .$$

Denote by $c^\pm(t,s), h^\pm(t,s)$ the solutions of (19), (20) with initial values $c_0^\pm(s), h_0^\pm(s)$. Those solutions are nonnegative and

$$c \ = \ c^+ - c^-, \qquad h \ = \ h^+ - h^- \, .$$

Conservation of bills implies that

$$\int_0^\infty c^\pm(t,s) \ + \ h^\pm(t,s) \ ds \ = \ \int_0^\infty c^\pm(0,s) + h^\pm(0,s) \ ds \, .$$

Therefore

$$\begin{aligned}
\int_0^\infty |c(t,s)| + |h(t,s|\, ds \ &\leq \ \int_0^\infty c^+(t,s) + c^-(t,s) + h^+(t,s) + h^-(t,s) \ ds \\
&= \ \int_0^\infty c^+(0,s) + c^-(0,s) + h^+(0,s) + h^-(0,s) \ ds \\
&= \ \int_0^\infty |c(0,s)| + |h(0,s)| \ ds \, .
\end{aligned}$$

To prove **ii.** use the same notations as in **i.**. If c, h are not everywhere of one sign, then there are points $s^\pm$ where $c^\pm(0,s^\pm) + h^\pm(0,s^\pm) > 0$. Proposition

9.ii implies that $c^\pm$ and $h^\pm$ are strictly positive on the domain of influence $D^\pm$ of $(0, s^\pm)$.

Choose t so large that the two domains $D^\pm$ overlap so there is an interval on which both c^+ and c^- are strictly positive and h^+ and h^- are strictly positive For such t

$$\int_0^\infty |c(t,s)|\ ds \ = \ \int_0^\infty |c^+(t,s) - c^-(t,s)|\ ds \ < \ \int_0^\infty c^+(t,s) + c^-(t,s)\ ds,$$

and

$$\int_0^\infty |h(t,s)|\ ds \ = \ \int_0^\infty |h^+(t,s) - h^-(t,s)|\ ds \ < \ \int_0^\infty h^+(t,s) + h^-(t,s)\ ds,$$

so the inequality in the proof of **i** is strict. ∎

5.2 Equilibria and pointwise bounds.

The equilibria or steady state solutions $(c(s), h(s))$ are the solutions of

$$\partial_s c \ = \ -a\,c + b\,h - \delta(s)\,c\,, \qquad 0 \ = \ a\,c - b\,h\,, \tag{23}$$

$$c(0) \ = \ \int_0^\infty \delta(s)\,c(s)\ ds\,. \tag{24}$$

Using the second equation in (23) in the first yields,

$$\partial_s c \ = \ -\delta(s)\,c(s)\,. \tag{25}$$

Integrate (25) from $s = 0$ to $s = \infty$ yields

$$-\int_0^\infty \delta(s)\,c(s)\ ds \ = \ \int_0^\infty \partial_s c(s)\ ds \ = \ c(\infty) - c(0) \ = \ -c(0)\,,$$

showing that (24) is a consequence of (25).

The general solution of (25) is found by the classical computation

$$\big(\ln c(s)\big)' \ = \ \frac{c'}{c} \ = \ -\delta(s). \tag{26}$$

Define for $s \geq 0$,

$$\alpha(s) \ := \ \int_0^s \delta(\tau)\ d\tau\,.$$

Integrating (26) from $s = 0$ yields

$$\ln c(s) - \ln c(0) \;=\; -\alpha(s)\,,$$

whence

$$c(s) \;=\; A\, e^{-\alpha(s)}\,, \qquad A = c(0)\,.$$

Then $h(s) = ac(s)/b$ and the total number of bills is

$$N \;=\; A\ (1 + a/b)\ \int_0^\infty e^{-\alpha(s)}\ ds\,.$$

For each number of bills, N, there is a unique value of A and therefore a unique steady state with that number of bills.

A difference between the $\delta(s)$ model and the model in §2 is that the steady state has an age dependent population density. If one measured in the steady state, the histogram of bill ages, that would yield the curve $e^{-\alpha(s)}$ and therefore $\alpha(s)$. Differentiation would yield $\delta(s)$.

The steady state histogram of bill ages determines $\delta(s)$ and depends on neither a or b.

The next result uses the equilibria to obtain an upper bound on solutions uniformly in time.

Proposition 11 i. *If $c(t,s), h(t,s)$ is a solution of (19), (20) and $\tilde{c}(s), \tilde{h}(s)$ is an equilibrium of that system with the property that*

$$|c(0,s)| \;\le\; \tilde{c}(s)\,, \qquad \text{and} \qquad |h(0,s)| \;\le\; \tilde{h}(s)\,,$$

then, for all $t \ge 0$ one has

$$|c(t,s)| \;\le\; \tilde{c}(s)\,, \qquad \text{and} \qquad |h(t,s)| \;\le\; \tilde{h}(s)\,.$$

ii. *If $c(t,s), h(t,s)$ is a continuously differentiable solution of (19), (20) with initial data vanishing for s large, then there is an equilibrium solution $\tilde{c}(s), \tilde{h}(s)$ so that for all $t \ge 0$ and $s \ge 0$*

$$|c(t,s)| + |c_t(t,s)| + |c_s(t,s)| + |h(t,s)| + |h_t(t,s)| + |h_s(t,s)| \;\le\; \tilde{c}(t,s)\,.$$

Proof. To prove **i**, define $c^{\pm}(t,s), h^{\pm}(t,s)$ as in the proof of Proposition 10. The difference $\tilde{c}-c^{\pm}, \tilde{h}-h^{\pm}$ is a solution of the $\delta(s)$ model with nonnegative initial data. It follows that the solution remains nonnegative for all $t \geq 0$ that is

$$0 \;\leq\; c^{\pm}(t,s) \;\leq \tilde{c}\,, \qquad 0 \;\leq\; h^{\pm}(t,s) \;\leq \tilde{h}\,.$$

Since $c^{\pm}$ are nonnegative and $c = c^+ - c^-$, it follows that

$$|c| \;\leq\; \max\{c^+, c^-\} \;\leq\; \tilde{c}\,,$$

with a similar argument for $|h|$. The proof of **i** is complete.

To prove **ii**, observe that if f $(c(t,x), h(t,s))$ is a solution, then so is $\big(\partial_t c, \partial_t h\big)$. Choose an equilibrium $\tilde{c}, \tilde{h}$ so that

$$|c_t(0,s)| \;\leq \tilde{c}(s)\,, \qquad \text{and} \qquad |h_t| \;\leq\; \tilde{h}(s)\,.$$

Part **i** shows that these upper bounds remain valid for $t \geq 0$.

Once $|c|$ and $|c_t|$ and $|h|$ are bounded above by equilibria, the same is true for c_s since the first of the equations (19) expresses c_s as a combination of the other three.

To estimate the derivative h_s, differentiate (21) to find

$$\partial_t h_s \;=\; a\,c_s - b\,h_s\,, \qquad h_s(t,s) \;=\; e^{-tb}\,h_s(0,s) + \int_0^t e^{-(t-s)b}\; a\,c_s(t,s)\;dt\,.$$

The integrability of e^{-bt} yields the desired bound for h_s. ∎

5.3 Behavior of the bulk quantities.

Recall equation (3). Integrating the dynamic equation for c yields

$$\int_0^\infty \Big(\partial_t c+\partial_s c\Big)\,ds \;=\; \int_0^\infty \Big(-ac+bh-\delta c\Big)\,ds \;=\; -aC+bH-\int_0^\infty \delta(s)c(t,s)\,ds\,.$$

Leibniz' rule for differentiating under the integral shows that

$$\int_0^\infty \partial_t c\;ds \;=\; \partial_t \int_0^\infty c(t,s)\;ds \;=\; C'(t)\,.$$

The Fundamental Theorem of Calculus implies that

$$\int_0^\infty \partial_s c(t,s)\ ds = c(\infty) - c(0) = -c(0)\,.$$

Using these identities in the integrated dynamic law and using also the boundary condition yields

$$C' = -aC + bH\,.$$

A similar but simpler calculation yields

$$H' = aC - bH\,.$$

This is exactly the system of ordinary differential equations in §2. The large time behavior of that equation yields the following result.

Proposition 12 *As $t \to \infty$ one has*

$$C(t) \to \frac{b}{a+b}\,N\,, \qquad H(t) \to \frac{a}{a+b}\,N\,,$$

where N is the the total number of bills. The convergence is at the exponential rate $e^{-(a+b)t}$.

It is remarkable that this conclusion holds *for all removal rates* $\delta(s)$. The evolution of the bulk quantities is not affected by δ. Measuring the rate at which the bulk quantity $C(t)$ relaxes to equilibrium yields $a+b$. The ratio b/a determines and is determined by the relative size of C and H. In practice the two bulk quantities are of comparable size, the coefficients a, b are of comparable size.

5.4 Large time behavior.

In this section, the preceding estimates together with the strategy of Lasalle's Invariance Principle shows that solutions of the $\delta(s)$ model tend for large time to the equilibrium of that model which has the same total number of bills.

Theorem 13 *Suppse that c, h is a continuously differentiable solution of (19), (20) whose initial data vanish for s large. Denote by c_e, h_e the equilibrium solution with the same number of bills,*

$$\int_0^\infty c(t,s) + h(t,s)\, ds = \int_0^\infty c_e(s) + h_e(s)\, ds\,.$$

Then as $t \to \infty$, c, h converges to c_e, h_e in the sense that

$$\lim_{t\to\infty} \quad \int_0^\infty |c(t,s) - c_e(s)| + |h(t,s) - h_e(s)|\; ds \;=\; 0\,,$$

and

$$\lim_{t\to\infty} \quad \max_{0\le s<\infty} \Big(|c(t,s) - c_e(s)| + |h(t,s) - h_e(s)| \Big) \;=\; 0\,,$$

Proof. Denote by

$$\underline{c} \;:=\; c - c_e\,, \qquad \underline{h} \;:=\; h - h_e\,,$$

the solution which is the deviation from equilibrium. Apply Proposition 11.**ii** to find an equilibrium $\tilde c, \tilde h$ so that $|\underline{c}|, |\underline{c}_t|, |\underline{c} - s|, |h|, |h_t|, |h_s|$ are all bounded above by $\tilde c$.
Proposition 10.**i** implies that

$$\int_0^\infty |\underline{c}(t,s)| + |\underline{h}(t,s)|\; ds$$

is a nonincreasing nonnegative function of t. Denote by $E \ge 0$ its limit as $t \to \infty$. The first assertion of the Theorem is that $E = 0$. So we need to show that $E > 0$ is impossible.
Choose $t_n > 0$ so that

$$E \;\le\; \int_0^\infty |\underline{c}(t_n,s)| + |\underline{h}(t_n,s)|\; ds \;\le\; E + \frac{1}{n}\,.$$

Then the sequence of solutions,

$$\big(c_n(t,s), h_n(t,s)\big) \;:=\; \big(\underline{c}(t+t_n,s), \underline{h}(t+t_n,s)\big)$$

inherits the pointwise upper bounds of $\underline{c}, \underline{h}$, and, for all $t \ge 0$,

$$E \;\le\; \int_0^\infty |c_n(t,s)| + |h_n(t,s)|\; ds \;\le\; E + \frac{1}{n}\,.$$

In addition, since $\underline{c}, \underline{h}$ has zero total bills, one has

$$\int_0^\infty c_n(t,s) + h_n(t,s)\ ds \ = \ 0\,.$$

Ascoli's Theorem asserts that there is a subsequence of the c_n, h_n which converges uniformly on compact subsets of $[0,\infty[\times[0,\infty[$ to a uniformly lipshitz continuous solution of the $\delta(s)$ model. The limit inherits the pointwise bounds for $\underline{c}, \underline{h}$ and its first derivatives. In addition, the dominated convergence theorem allows one to pass to the limit in the preceding two estimates to show that for all $t \geq 0$,

$$\int |c(t,s)| + |h(t,s|\ ds \ = \ E\,, \quad \text{and} \quad \int c(t,s) + h(t,s)\ ds \ = \ 0\,.$$

If $E > 0$, then the vanishing total number of bills shows that it is *not* true that c and h have the same sign. Proposition 10.**ii.** shows that in this case it is impossible to have $\int |c| + |h|\, ds$ independent of $t \geq 0$. The conclusion is that $E = 0$ which completes the proof of the first assertion of the Theorem.

The second, assertion of the Theorem is a consequence of the first. To see this let

$$F(t) \ := \ \max_{0 \leq s < \infty} \Big(|c(t,s) - c_e(s)| + |h(t,s) - h_e(s)| \Big) \ \geq 0\,.$$

For $t > 0$ choose $\underline{s} = \underline{s}(t)$ so that

$$|c(t,\underline{s}) - c_e(\underline{s})| + |h(t,\underline{s}) - h_e(\underline{s})| \ \geq \ F(t)/2\,.$$

Choose $K > 0$ so that for all $t \geq 0$ and $s \geq 0$,

$$\big|\partial_s(c(t,s) - c_e(s))\big| + \big|\partial_s(h(t,s) - h_e(s))\big| \ \leq \ K\,.$$

Then, on the interval

$$|s - \underline{s}| \ < \ \frac{F(t)}{4K}\,,$$

one has

$$|c(t,s) - c_e(s)| + |h(t,s) - h_e(s)| \ \geq \ F(t)/4\,.$$

Therefore

$$\begin{aligned}
\int_0^\infty |c(t,s) - c_e(s)| + |h(t,s) - h_e(s)| \; ds \\
&\geq \int_{|s-\underline{s}|<F/4K} |c(t,s) - c_e(s)| + |h(t,s) - h_e(s)| \; ds \\
&\geq 2\, \frac{F(t)}{4K}\, \frac{F(t)}{4}\,.
\end{aligned}$$

The first assertion of the Theorem shows that left hand side tends to zero as $t \to \infty$. It follows that $F(t)$ tends to zero which is the second assertion of the Theorem. ∎

6 Growth and inflation.

We study models where more bills are replaced than are removed. The total number of bills grows. This is the model which best fits normal monetary policy, and observed data. We make the realistic assumption that there is a finite upper limit S, on the number of years a bill can spend in circulation. The resulting model takes place on a finite interval $0 \leq s \leq S$,

$$c_t + c_s + a\,c - b\,h + \delta(s)\,c \;=\; 0\,, \qquad 0 \leq s \leq S\,, \tag{27}$$

$$h_t - a\,c + b\,h \;=\; 0\,, \quad 0 \leq s \leq S\,, \tag{28}$$

$$h(t,0) = 0\,. \tag{29}$$

The replacement of bills is handled with a boundary condition

$$c(t,0) \;=\; (1+r) \int_0^S \delta(s)\, c(s)\; ds\,, \qquad 0 \leq r\,. \tag{30}$$

The model posits that bills which are removed from circulation are replaced with an inflationary factor $1+r$ while those that circulate for S years simply disappear from circulation. If S is chosen reasonably large, there will be very few. On a technical level admitting the possibility of arbitrarily old bills poses mathematical problems which are irrelevant in the practical situation where S less than 20 is more than reasonable from the data in §2.

The total number of bills satisfies

$$\partial_t \int_0^S c(t,s) + h(t,s)\ ds \ = \ -c(t,S) \ + \ r \int_0^S \delta(s)\ c(t,s)\ ds\,.$$

6.1 Exponentially growing modes.

Seek solutions of the form

$$e^{\gamma t}(c(s), h(s))\,. \tag{31}$$

The differential equations are equivalent to,

$$(\gamma + \partial_s)c + ac - bh + \delta(s)\,c = 0\,, \qquad \gamma h - ac + bh \ = \ 0\,. \tag{32}$$

The second yields

$$ac - bh = \gamma h = \frac{\gamma a}{b+\gamma}\,c\,.$$

Plugging in yields

$$\partial_s c + \Big(\rho(\gamma) + \delta(s)\Big)c \ = \ 0\,, \qquad \rho(\gamma) \ := \ \gamma + \frac{\gamma a}{b+\gamma}\,.$$

With $\alpha(s)$ as in Section 5.2, the solutions $c(s)$ are constant multiples of

$$c(s) \ = \ e^{-\rho(\gamma)s - \alpha(s)}\,. \tag{33}$$

The boundary condition is satisfied if and only if

$$\frac{1}{1+r} \ = \ \int_0^S \delta(s)\ e^{-\rho(\gamma)s-\alpha(s)}\ ds\,. \tag{34}$$

The function $\rho(\gamma)$ is strictly monotone increasing for $\gamma \geq 0$ so the right hand side is a decreasing function of $\gamma \geq 0$ which tends to 0 as $\gamma \to \infty$.
Define r_0 by

$$\frac{1}{1+r_0} \ := \ \int_0^S \delta(s)\ e^{-\alpha(s)}\ ds\,.$$

For $r = r_0$, the boundary condition is satisfied for $\gamma = 0$.
Given $r > r_0$ there is exactly one value of $\gamma > 0$ so that the boundary condition is satisfied. It is for this range of r that the number of circulating bills is growing. These computations prove the following result.

Proposition 14 *For each $r > r_0$ there is only one value of $\gamma > 0$ so that the boundary value problem (32) with boundary condition (30) has nontrivial solutions. For each such r there is a unique solution γ of (34). The function c is equal to a constant multiple of (33), and h is given in terms of c by $h = ac/(b + \gamma)$.*

6.2 Large time behavior.

As in all the models we have studied, nonnegative initial data lead to nonnegative solutions and therefore there is a comparison principle which asserts that for two solutions, one has

$$c_1(t,s) \ \le \ c_2(t,s)\,, \qquad \text{and} \qquad h_1(t,s) \ \le \ h_2(t,s)\,, \qquad 0 \le s \le S$$

for all $t \ge 0$, if and only if the inequalities holds at $t = 0$.

If $(c(0,s), h(0,s))$ is a bounded nonnegative data set, then one can choose $M > 0$ sufficiently large so that

$$c(0,s) \ \le \ M\,c(s) \qquad \text{and} \qquad h(0,s) \ \le \ M\,h(s)\,.$$

The order relations remain true for all time so

$$c(t,x) \ \le \ M\,e^{\gamma t} c(s) \qquad \text{and} \qquad h(t,s) \ \le \ M\,e^{\gamma t} h(s)\,.$$

This shows that solutions can grow no faster than $e^{\gamma t}$.

Theorem 15 *Suppose that $r > r_0$, γ, and $(c(s), h(s))$ define the exponentially growing mode from Proposition 14. Then there is a $\gamma' < \gamma$ and $C > 0$ so that if $c(t,s), h(t,s)$ is a nonnegative solution, there is a constant A so that*

$$\|(c(t,s),h(t,s)) - Ae^{\gamma t}(c(s),h(s))\|_{L^\infty([0,S])} \le C\,e^{\gamma' t}\,\|(c(0,s),h(0,s))\|_{L^\infty([0,S])}\,.$$

Proof of Theorem 15. Introduce the mapping $R(t)$ which sends the value of a solution at time $t = 0$ to the solution at time $t \ge 0$. The Riemann Matrix $R(t,s,\sigma)$ is the Schwartz kernel of this map defined by,

$$(c(t,s), h(t,s)) \ = \ \int R(t,s,\sigma)\big(c(0,\sigma), h(0,\sigma)\big)\,d\sigma\,.$$

For $\sigma \in]0, S[$ fixed, each column of the 2×2 matrix R satisfies the equations (27), (28), together with the boundary condition (30). The first (resp. second) column has initial value equal to $(\mu_\sigma\,,\, 0)$ (resp. $(0\,,\,\mu_\sigma)$) where μ_σ is a unit point mass (a.k.a. Dirac delta) supported at σ. Computations dating to the original work of Riemann show that for $0 \le t < S - \sigma$

$$R(t, s, \sigma) \;=\; \begin{pmatrix} e^{-at}\mu_{\sigma+t} & 0 \\ 0 & e^{-bt}\mu_\sigma \end{pmatrix} \quad \mathrm{mod} \;\; L^\infty\,.$$

When the point mass moving to the right encounters the boundary it disappears into the boundary. The boundary condition induces a jump discontinuity in c to emerge from $s = 0$ at that time. For $t > S$ the difference

$$F \;=\; R(t, s, \sigma) \;-\; \begin{pmatrix} 0 & 0 \\ 0 & e^{-bt}\mu_\sigma \end{pmatrix}$$

is piecewise smooth function of t, s uniformly bounded on $\{S < t \le T\} \times \{0 \le s \le S\} \times \{0 < \sigma < S\}$. This can be proved by constructing, using the method of progressing waves as in [2], [4], corrections to this leading term which yield approximate solutions whose error is as smooth as one likes. Thus, the operator $R(t)$ mapping initial data to data at time t is for $t > S$ the sum of $\mathrm{diag}\big(0\,,\, e^{-bt}\big)$ and a compact operator with bounded piecewise continuous kernel.

Fix $t > S$. Consider $R(t)$ as a map from $L^\infty([0, S])$, $R(t)$ to itself. The preceding result shows that the intersection of the spectrum of $R(t)$ with $\big\{|z| > e^{-bt}\big\}$ consists of a discrete set of eigenvalues of finite multiplicity.

Choose $r \in]e^{-bS}, 1[$ and denote by $\mathbb{V}$ the finite dimensional complex vector space spanned by the generalized eigenspaces corresponding to eigenvalues λ with $|\lambda| \ge r$. The space $\mathbb{V}$ contains the exponentially growing real mode $(c(s), h(s))$ from Section 6.1 corresponding to the positive eigenvalue $e^{\gamma t}$ of $R(t)$.

Denote by $\mathbb{P}$ the open cone of functions (c, h) so that the essential infinium of both c and h are strictly positive. The operators $R(t)$ map $\mathbb{P}$ to itself. Define the finite dimensional real vector space

$$\mathbb{W} \;:=\; \mathrm{Re}\,\mathbb{V}\,.$$

Then $\mathbb{W}$ contains the strictly positive growing mode $(c(s), h(s))$ and is invariant under $R(t)$ for all t. $\mathbb{P} \cap \mathbb{W} \ni (c(s), h(s))$ is a nonempty relatively open cone in $\mathbb{W}$ endowing $\mathbb{W}$ with the structure of an ordered vector space. The Perron-Frobenius Theorem (see [3] for example) applied to $R(t)\big|_{\mathbb{W}}$, the restriction of R to $\mathbb{W}$, shows that the largest eigenvalue of $R(t)$ is $e^{\gamma t}$ and has algebraic and geometric multiplicity equal to one. The other eigenvalues are strictly smaller in absolute value.

Denote by Π the spectral projection of $R(1)$ on the principal eigenspace

$$\Pi \;:=\; \frac{1}{2\pi i}\oint_{|z-e^{\gamma}|=\rho} \big(R(1)-zI\big)^{-1}\,dz, \qquad 0<\rho<<1.$$

Then

$$\mathbb{V} \;=\; \mathrm{Range}(I-\Pi)\;\oplus\;\mathbb{C}\left(c(s),h(s)\right).$$

The Perron-Frobenius argument shows that there is a $0<\widetilde{\gamma}<\gamma$ so that the restriction of $R(1)$ to $\mathrm{Range}(I-\Pi)$ has spectrum in $\{|z|\le\widetilde{\gamma}\}$. The spectral radius formula implies that for any $\gamma'\in]\widetilde{\gamma},\,\gamma[$ there is a C so that for all $v\in\mathrm{Range}(I-\Pi)$ and $t\ge 0$,

$$R(t)v \;\le\; Ce^{\gamma' t}\|v\|\,.$$

Together with $R(t)\big(c(s),h(s)\big)=e^{\gamma t}\big(c(s),h(s)\big)$ this completes the proof. ∎

6.3 Conclusions.

• Once a policy with parameters r,δ is in place for a while, the bills will settle into the exponentially growing mode $A\,e^{\gamma t}(c(s),h(s))$. The parameter γ is the growth rate of the number of bills in circulation so is observable.

• Given the growth rate, γ_1 for one dollar bills, the known age histogram for one dollar bills determines $\rho(\gamma_1)s+\alpha(s)$. For one dollar bills, hoarding is irrelevant so $a=b=0$ yielding $\rho(\gamma_1)=\gamma_1$. Thus, the age histogram determines $\gamma_1 s+\alpha(s)$ and therefore $\alpha(s)$ and $\delta(s)$.

• For higher denominations it is reasonable that $\delta(s)$ is the same or similar to its value for one dollar bills.

• The age histogram for one dollar notes predicts $\rho(\gamma)$. Since γ is known this gives $a/(b+\gamma)$ which is the ratio c/h in the steady state. Thus the hoarded population would be estimated.

• One can find r from (34).

• One does not yet have a, b. For the trio a, b, γ one would know γ and $a/(b+\gamma)$. *One more measurement is needed to get a and b.* One gets an estimate for the hoarded bills without this additional measurement.

• The missing datum could be extracted from the age histogram of the circulating bills in the higher denomination. If the model is correct, the histogram will be different from that of one dollar bills because of hoarding.

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