The Time Integrated Far Field for Maxwell's and D'Alembert's Equations

Jeffrey RAUCH[1]
Department of Mathematics
University of Michigan, Ann Arbor MI, USA
rauch@umich.edu

Gérard MOUROU [2]
Department of Electrical Engineering and Computer Science
University of Michigan, Ann Arbor MI, USA
mourou@umich.edu

Abstract. For x large consider the electric field, $E(t,x)$ and its temporal Fourier Transform, $\hat{E}(\omega, x)$. The D.C. component $\hat{E}(0,x)$ is equal to the time integral of the electric field. Experimentally, one observes that the D.C. component is negligible compared to the field. In this paper we show that this is true in the far field for all solutions of Maxwell's equations. It is not true for typical solutions of the scalar wave equation. The difference is explained by the fact that though each component of the field satisfies the scalar wave equation, the spatial integral of $\partial_t E(t,x)$ vanishes identically. For the scalar wave equation the spatial integral of $\partial_t u(t,x)$ need not vanish. This conserved quantity gives the leading contribution to the time integrated far field. We also give explit formulas for the far field behavior of the time integrals of the intensity.

2000 Mathematical Subject Classification. 78M35, 78A40, 78A60, 35L05.
Keywords and phrases. farfield, Maxwell's equations, wave equation, d.c. component.
[1] Partially supported by the US National Science Foundation grant NSF-DMS-0104096.
[2] Partially supported by the US National Science Foundation grant NSF-PHY-0114336.

§0. Introduction

The goal of this paper is to explain with a rigourous proof the observed fact that regardless of the initial state, the electric field in the far field has vanishing D.C. component. For terahertz lasers, such field strenths can be measured as functions of time using using ultrashort lasers whose pulse lengths are small compared to the variations in the field of the much broader terahertz lasers. Compared to a femtosecond laser, the field of a terahertz laser varies slowly.

Represent $E(t,x)$ for x in the far field in Fourier by

$$E(t,x) \;=\; \int_0^\infty a(\omega)\, e^{i\omega t} + a(-\omega)\, e^{-i\omega t}\ d\omega\,.$$

A typical experimental plot of measured $E(t,x)$ against t, and of $\big(|a(\omega)|^2 + |a(-\omega)|^2\big)^{1/2}$ against frequency $\omega \geq 0$ is given in Figure 1 taken from [1]. The Fourier transform always passes through the origin. The D.C. component is negligible compared to typical components of the fields.

Figure 1. The second graph shows vanishing D.C. component

An intuitive but incorrect explanation of this is the following. The D.C. component has zero wave number and therefore does not have a natural direction of propagation. It therefore does not propagate. The fact that this argument is not correct is demonstrated by the observation that it applies equally well to the scalar wave equation and to Maxwell's Equations. However the vanishing D.C. conclusion is false for the former and true for the latter. An experiment as in Figure 1 performed on acoustic waves would normally yield a spectrum which does not pass through the origin.

A correct intuitive argument is the following. For wave number equal to zero, propagation is at the speed of light with all directions of propagation equally likely. The spatial Fourier transform evaluated at frequency zero of the initial data spreads out in all directions. Other contributions oscillate in time and therefore have small time integrals. The time integrated field is dominated by the component of the initial data. For Maxwell's equations, the condition div$E = 0$ means that $\hat{E}(t,\xi)$ is perpendicular to ξ. Since in a neighborhood of $\xi = 0$, ξ points in all directions, it follows that if $\hat{E}$ is continuous at $\xi = 0$ it must vanish there. Thus, for Maxwell's equations, the $\xi = 0$ component of the data vanishes. Thus, it is reasonable that the time integrated field is weak. Our main results make this precise and distinguishes clearly between the scalar wave equation and Maxwell's equations.

In order to concentrate on essentials we treat the case of infinitely smooth compactly supported solutions. A crude sufficient involving only finite regularity and decay at infinity is noted at the end of §3. We treat only the case $d = 3$ of experimental interest.

Theorem 1. *If $u(t,x)$ satisfies D'Alembert's wave equation,*

$$\Box u = 0\,, \qquad \Box := \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\,, \tag{1}$$

with Cauchy data $u(0) := u(0,x)$ and $u_t(0,x) := \partial u(0,x)/\partial t$, in $C_0^\infty(\mathbb{R}^3)$, then the time integrated far field satisfies

$$\int_0^\infty u(t,x)\,dt = \frac{1}{4\pi|x|}\int_{\mathbb{R}^3} u_t(0,y)\,dy - \frac{x}{4\pi|x|^3}\cdot\int y\,u_t(0,y)\,dy \;+\; O(1/|x|^3)\,, \quad \text{as } |x|\to\infty\,. \tag{2}$$

Remarks. i. Since the solution $u = O(1/|x|)$ in the far field, this shows that the time integral is comparable to the field strengths. **ii.** The leading term is independent of direction. Though the far field itself can be strongly directional, the time integral is not. **iii.** It is easily verified that formula (2) is correct on the explicit spherical wave solutions

$$u = \frac{f(t+r) - f(t-r)}{r} \qquad \text{with} \quad f\in C_0^\infty(\mathbb{R})\,,$$

for which

$$\int_0^\infty u(t,x)\;dt = \frac{1}{r}\int_{-\infty}^{\infty} f(s)\;ds + O(1/r^2)\,.$$

iv. The surprising fact that (2) involves $u_t(0)$ and not $u(0)$ is discussed in §2. **v.** The coefficients in the espanasion, $\int u_t(t,x)\,dx$ and $\int x\,u_t(0,x)\,dx$ are conserved quantities for solutions of the wave equation. **v.** D'Alembert's equation describes linear acoustic and also linear elastic waves.

In sharp contrast, the time averages for Maxwell's Equations are much smaller.

Corollary 2. *If E, B are solutions of Maxwell's equations*

$$\frac{\partial E}{\partial t} = \mathrm{curl} B\,, \qquad \frac{\partial B}{\partial t} = -\,\mathrm{curl} E\,, \qquad \mathrm{div} E \;=\; \mathrm{div} B \;=\; 0\,,$$

with $E(0,x), B(0,x) \in C_0^\infty(\mathbb{R}^3)$, then in the far field $E, B = O(1/|x|)$ and the time averages are negligible compared to the fields,

$$\int_0^\infty E(t,x)\,dt \;=\; O(1/|x|^3)\,, \qquad \int_0^\infty B(t,x)\,dt \;=\; O(1/|x|^3)\,, \qquad \text{as } |x| \to \infty\,. \tag{3}$$

The explanation of this difference is that, though the components of the electromagnetic field are solutions of D'Alembert's equation, the Maxwell equations imply that $\partial E/\partial t$ and $\partial B/\partial t$ are both sums of x derivatives of functions vanishing at infinity. The fundamental theorem of calculus implies that

$$\int_{\mathbb{R}^3} \frac{\partial E(0,y)}{\partial t}\,dy \;=\; \int_{\mathbb{R}^3} \frac{\partial B(0,y)}{\partial t}\,dy \;=\; 0\,,$$

so the leading term on the right hand side of (2) vanishes. Before Theorem 1, it was explained how the divergence free condition also leads to the decay result. In §2 we combine the two explanations to show that the second, $O(1/|x|^2)$, term also vanishes. We thank P. Tchamitchian for that refinement.

In §4, we also prove formulas for the time integrals of u^2. Such quadratic quantities are important as they correspond to the total energy deposited.

Theorem 3. *If u is as in Theorem 1 then for unit vectors Ω, define*

$$g(\Omega, y) \;:=\; u_t(0,y) - \Omega.\nabla_y u(0,y)\,,$$

and its Fourier transform with respect to y,

$$(\mathcal{F}g)(\Omega,\xi) \;:=\; (2\pi)^{-3/2} \int g(\Omega,y)\; e^{-iy.\xi}\;dy\,.$$

Then

$$\int_0^\infty u^2(t,x)\,dt \;=\; \frac{1}{4|x|^2} \int \Big|(\mathcal{F}g)(x/|x|, \alpha x/|x|)\Big|^2\,d\alpha \;+\; O(1/|x|^3)\,. \tag{4}$$

The main contribution is the integral of the squared modulus of a Fourier Transform over all frequencies, $\alpha\, x/|x|$, parallel to x.

§1. A short proof of (2) and (3).

This short proof has the defect of not explaining why (2) is true. It also does not extend to other quantities like (4).

Proof of Theorem 1. Suppose that the support of the Cauchy data is contained in the ball $|x| \le R$. Then the solution u is infinitely differentiable and for $t \ge 0$ is supported in the outgoing shell $\{(t,x)\,:\, t - R \le |x| \le t + R\}$.

Integrating the differential equation from $t = 0$ to $t = \infty$ and setting

$$A(x) \ := \ \int_0^\infty u(t,x)\ dt\,, \tag{5}$$

yields Poisson's equation

$$\Delta A(x) \ = \ -u_t(0,x)\,, \qquad \Delta \ := \ \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\,,$$

with source $-u_t(0,y)$.
The retarded potential formula ([2, §III.4.2]) for the solution of D'Alembert's equation implies that

$$|u(t,x)| \ \le \ \frac{C}{1+|x|}\,.$$

It follows that $A = O(1/|x|)$ as $x \to \infty$. Therefore, A is given by Newton's formula for the solution of Poisson's equation [2,§IV.1.2],

$$A(x) = \frac{1}{4\pi} \int \frac{u_t(0,y)}{|x-y|}\ dy\,.$$

The multipole approximation for y in the support of $u_t(0,y)$ and $x \to \infty$ yields

$$\frac{1}{|x-y|} \ = \ \frac{1}{|x|}\,\frac{1}{\big|x/|x| - y/|x|\big|} \ = \ \frac{1}{|x|}\Big(1 - \frac{x.y}{|x|^2} + O(y/|x|^2)\Big) \ = \ \frac{1}{|x|} - \frac{x.y}{|x|^3} + O(1/|x|^3)\,.$$

Therefore,

$$A(x) = \frac{1}{4\pi|x|}\int u_t(0,y)\ dy \ - \ \frac{x}{4\pi|x|^3}\cdot\int y\,u_t(0,y)\ dy \ + \ O(1/|x|^3)\,, \qquad \text{as} \quad x \to \infty\,. \tag{6}$$

This is formula (2). ∎

Proof of Corollary 2. Thanks to (6), to verify (3) it suffices to show that

$$\int y.\,\mathrm{curl}\,B\,dy = \int y.\,\mathrm{curl}\,E\,dy = 0.$$

For the first note that the integral is a sum of terms of the form

$$\pm\int_{\mathbb{R}^3} y_j\,\frac{\partial B}{\partial y_k}\ dy\,, \quad \text{with} \quad j \ne k\,.$$

Since $j \ne k$, an integration by parts shows that the integral vanishes. The same argument works for the $\mathrm{curl}\,E$ integral. ∎

§2. Why $u(0,x)$ does not appear in (2).

It is surprising that formula (2) involves $\partial_t u(0,x)$ and not $u(0,x)$. To understand why $u(0,x)$ does not appear, define v and w to be the solutions of

$$\Box v \;=\; \Box w = 0\,,$$

$$v(0,x)=0, \quad v_t(0,x)=u_t(0,x)\,, \qquad w(0,x)=0, \quad w_t(0,x)=u(0,x)\,.$$

Then

$$u \;=\; v \;+\; \frac{\partial w}{\partial t}$$

since the right hand side is a solution of D'Alembert's equation with the same Cauchy data as u. When $|x|>R$, $w(t,x)$ vanishes for $0\le t\le |x|-R$ and for $t>|x|+R$ so the fundamental theorem of calculus implies that

$$\int_0^\infty \frac{\partial w}{\partial t}(t,x)\;dt \;=\; 0\,.$$

Therefore the time integral of u depends only on v. Thus, the time integral is independent of $u(0,x)$.

§3. A derivation which explains.

For $t>0$, the solution v of the preceding section is given by the retarded potential formula ([2,§III.4.2],

$$v(t,x) \;=\; \frac{1}{4\pi t}\int_{|x-y|=t} u_t(0,y)\,d\sigma(y)\,, \tag{7}$$

where $d\sigma$ denotes the element of area on the sphere $|x-y|=t$. When the initial data vanishes for $|y|\ge R$ this integral vanishes unless $-R<|x|-t<R$.

To evaluate $v(t,x)$ one integrates over the sphere $|x-y|=t$. Integrating in time yields spheres with common center x. One sums (integrates) the resulting integrals over the spheres. The geometry is sketched in Figure 2.

Figure 2. Retarded spheres sweep out $|y|\le R$.

When $|x|$ is large then the retarded potential formula (7) is nonzero only when $|x|$ and t differ by at most R so $1/t$ satisfies $1/(|x|+R)\le 1/t\le 1/(|x|-R)$ so $1/t=1/|x|+O(1/|x|^2)$.

For t large, the sphere $|x-y|=t$ has principal curvatures equal to $1/t=O(1/|x|)$ which tend to zero. Since the support of $u_t(0)$ is bounded, the sphere and the plane,

$$\Big\{y\;:\;\frac{x}{|x|}\cdot(x-y)=t\Big\} \;=\; \Big\{y\;:\;y\cdot\frac{x}{|x|}=|x|-t\Big\}$$

are nearly identical on the support of $u_t(0)$. Precisely, they and their tangent planes differ by quantities $O(1/|x|)$. Thus integrating with respect to surface area on the spheres followed by a summation over the spheres converges to an integration with respect to the Euclidean volume dy. The difference is $O(1/|x|)$. This yields

$$\int_0^\infty v(t,x)\ dt \ =\ \frac{1}{4\pi|x|}\int_{\mathbb{R}^3} u_t(0,y)\ dy\ +\ O(1/|x|^2)\,, \tag{8}$$

which together with the result of §2 gives a second proof of the leading term in formula (2).

It is not hard to show that (8) holds if at time $t=0$, u has continuous partial derivatives of order less than or equal to two which decay at infinity as fast as $1/(1+|y|)^{3+\delta}$ for some $\delta>0$.

§4. The time integral of u^2.

Integrals of this type, and most particularly $\int_0^\infty E^2+B^2\,dt$ are of interest because of their relation to energy deposited and also because they allow no possibility of cancellation since the integrand is nonnegative.

Proof of Theorem 3. In this proof, the functions v, w are as in §2. For unit vectors $\Omega\in\mathbb{R}^3$ and $s\in\mathbb{R}$, introduce the integrals with respect to surface area $d\sigma(y)$ on the planes $\{y.\Omega=s\}$,

$$m(\Omega,s)\ :=\ \frac{1}{4\pi}\int_{y.\Omega=s} u_t(0,y)\ d\sigma(y)\,.$$

The calculation of §3 shows that

$$v(t,x)\ =\ \frac{m\big(x/|x|\,,\,|x|-t\big)}{|x|}\ +\ O(1/|x|^2)\,. \tag{9}$$

This expresses the asymptotics of v as a outgoing spherical wave which is modulated in the angular directions. It is not hard to justify differentiating with respect to time to find

$$v_t(t,x)=\frac{-m_s\big(x/|x|\,,\,|x|-t\big)}{|x|}+O(1/|x|^2)\,, \tag{10}$$

where

$$m_s(\Omega,s)\ :=\ \frac{\partial m}{\partial s}=\frac{1}{4\pi}\int_{y.\Omega=s}\Omega.\nabla_y u_t(0,y)\ d\sigma(y)\,.$$

These formulas applied to w yield

$$\frac{\partial w}{\partial t}=\frac{n\big(x/|x|\,,\,|x|-t\big)}{|x|}+O(1/|x|^2)\,, \tag{11}$$

where

$$n(\Omega,s)\ :=\ \frac{-1}{4\pi}\int_{y.\Omega=s}\Omega.\nabla_y u(0,y)\ d\sigma(y)\,.$$

Adding equations (9) and (11) yields Friedlander's formula [3],

$$u(t,x)=\frac{L\big(x/|x|\,,\,|x|-t\big)}{|x|}+O(1/|x|^2)\,, \tag{12}$$

where

$$L(\Omega, s) \; := \; \frac{1}{4\pi} \int_{y.\Omega=s} u_t(0,y) - \Omega.\nabla_y u(0,y) \; d\sigma(y) \, .$$

Formula (12) yields

$$\int_0^\infty u^2(t,x) \; dt \; = \; \frac{1}{16\pi^2 |x|^2} \int_{-\infty}^\infty L(x/|x| \, , \, s)^2 \; ds + O(1/|x|^3) \, . \tag{13}$$

This equation has a nice form in terms of the Fourier Transform of the initial data. Consider the case of $x/|x| = \Omega = (1,0,0) := \mathbf{e}_1$ where

$$L(\mathbf{e}_1, s) = \frac{1}{4\pi} \int u_t(0, s, x_2, x_3) - u_{x_1}(0, s, x_2, x_3) \; dx_2 \, dx_3 .$$

Taking the Fourier Transform with respect to s yields

$$\begin{aligned} \hat{L}(\mathbf{e}_1, \alpha) \; &:= \frac{1}{\sqrt{2\pi}} \int L(\mathbf{e}_1, s) \, e^{-is\alpha} \, ds \\ &= \frac{2\pi}{(2\pi)^{3/2}} \int (u_t(0,x) - u_{x_1}(0,x)) e^{-ix_1\alpha} \, dx \\ &= 2\pi \, \mathcal{F}\big(u_t(0) - u_{x_1}(0)\big)(\alpha \mathbf{e}_1) \, . \end{aligned}$$

Parseval's identity implies that

$$\int |L(\mathbf{e}_1, s)|^2 \, ds = \int |\hat{L}(\mathbf{e}_1, \alpha)|^2 \, d\alpha = 4\pi^2 \int \Big| \mathcal{F}\big(u_t(0) - u_{x_1}(0)\big)(\alpha \mathbf{e}_1) \Big|^2 \; d\alpha \, .$$

Combining this with (13) and passing to general Ω yields (4). ∎

References

1) J.F. Whitaker, F. Gao, and Y. Liu, in *Nonlinear Optics for High-Speed Electronics and Optical Frequency Conversion*, N. Peygambarian, H. Everitt, R.C. Eckardt, D.D. Lowenthal eds., *Proc. SPIE*, vol. 2145, pp. 168-177 (1994)

2) R. Courant, *Methods of Mathematical Physics Volume II*, Interscience Publ. 1962.

3) F.G. Friedlander, On the radiation field of pulse solutions of the wave equation II, *Proc. Roy. Soc*, **279A**(1964), 386-394.