**Regularity of Functions Smooth Along Foliations,**

**And Elliptic Regularity**

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## 1. Introduction

Work on dynamical systems, particularly on Anosov systems, gives rise to foliations whose leaves are individually smooth but which vary in a much less smooth fashion. Basic constructions produce functions that are smooth along the leaves of one or more such foliations, and one wants to deduce further regularity properties of such functions. Notable results on such regularity are given in [J], [HK] and [dL]. It is our purpose to contribute further to such regularity results.

We consider the following type of foliation $\mathcal{F}$ of a smooth, compact, $n$-dimensional manifold $M$ by $m$-dimensional leaves. We assume that for each $p \in M$ there is a neighborhood $\mathcal{O}$ and a homeomorphism

$$\Gamma : U \times V \longrightarrow \mathcal{O}, \tag{1.1}$$

from a neighborhood $U \times V \subset \mathbb{R}^{n-m} \times \mathbb{R}^m$ of $0 \in \mathbb{R}^n$ onto $\mathcal{O}$, with $p = \Gamma(0,0)$, such that for each $x \in U$, $\{\Gamma(x,y) : y \in V\}$ is a leaf of $\mathcal{F}$ (restricted to $\mathcal{O}$). As for regularity of $\mathcal{F}$, we fix $r \in [0,\infty)$ and $k \in \mathbb{Z}^+ \cup \{\infty\}$, $k \ge 1$, and assume

$$\partial_y^\alpha \Gamma \in C^r(U \times V), \quad \forall\ |\alpha| \le k. \tag{1.2}$$

Here $C^0(U \times V)$ denotes the space of continuous functions on $U \times V$. If $r \in \mathbb{Z}^+$, $C^r(U \times V)$ denotes the space of functions with continuous derivatives up to order $r$. If $r = \ell + s$, $\ell \in \mathbb{Z}^+$, $0 < s < 1$, then $C^r(U \times V)$ denotes the space of functions whose derivatives of order $\ell$ are Hölder continuous of exponent $s$.

We furthermore assume that for each fixed $x$, the map $y \mapsto \Gamma(x,y)$ is a $C^k$ immersion of $V$ into $\mathcal{O}$. We also assume that $\Gamma$ is absolutely continuous in the sense that for Borel sets $E \subset U, E' \subset V$, the measure of $\Gamma(E \times E') \subset \mathcal{O}$ is given by

$$\int_E \int_{E'} J(x,y)\, dy\, dx, \tag{1.3}$$

and we assume $J > 0$ on $U \times V$ and

$$\partial_y^\alpha J \in C^r(U \times V), \quad \forall\ |\alpha| \le k. \tag{1.4}$$

If these conditions hold, we say the foliation $\mathcal{F}$ is regular of order $(k, r)$.

The case $r = 0$ will be the setting for most of our regularity results, as it was for such results in [HK] and [dL]. It is worth mentioning that the stable and the unstable foliation arising from a $C^{k+2}$ Anosov map are foliations that are regular of order $(k, r)$, for some $r > 0$. There are corresponding results for the weakly stable and the weakly unstable foliation arising from an Anosov flow. We refer to the introductory sections of [HK] and [dL] for more on this, including some of the history (proceeding from work of Anosov, Hirsch and Pugh, and others) and also more precise statements on the regularity of these foliations, which we do not need here. We do remark that situations arise for which one cannot take $r = 2$.

Given a function $u$ on $M$, regularity along a foliation $\mathcal{F}$ can be measured in terms of regularity of $u \circ \Gamma$. Here is a case of primary interest. Given $p \in (1, \infty)$, we say $u \in H^{k,p}_{\mathcal{F}}(M)$ provided $M$ is covered by open sets $\mathcal{O}$ and, for $\Gamma$ as described above,

$$\partial_y^\alpha (u \circ \Gamma) \in L^p(U \times V), \quad |\alpha| \leq k. \tag{1.5}$$

In terms of this notion, we can state Theorem 3 of [dL] as follows.

**Theorem 1.1.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_N$ be foliations of $M$, regular of order $(k, 0)$, with $k \geq 1$. Assume that, for each $x \in M$,*

$$T_x M = \mathit{Span}\{T_x \mathcal{F}_j : 1 \leq j \leq N\}. \tag{1.6}$$

*Then, given $p \in (1, \infty)$,*

$$u \in H^{k,p}_{\mathcal{F}_j}(M),\ \forall\, j \in \{1, \dots, N\} \Longrightarrow u \in H^{k,p}(M). \tag{1.7}$$

In (1.7), $H^{k,p}(M)$ is the standard $L^p$-Sobolev space of functions in $L^p(M)$ whose derivatives of order up to $k$ are in $L^p(M)$. Actually, Theorem 1.1 is a slight extension of Theorem 3 of [dL], which required $k$ to be even. In §2 we prove the following refinement of Theorem 1.1.

**Theorem 1.2.** *Let $\mathcal{F}$ be a foliation of $M$ that is regular of order $(k, 0)$, with $k \geq 1$. If $u \in H^{k,p}_{\mathcal{F}}(M)$, $p \in (1, \infty)$, then $u$ is in $H^{k,p}$ microlocally on any conic subset of $T^*M \setminus 0$ disjoint from the conormal bundle $\nu(\mathcal{F})$ to $\mathcal{F}$.*

We present an elementary description of the conclusion. Take $z_0 \in M$ and via a $C^\infty$ coordinate chart identify a neighborhood $\mathcal{O}$ of $z_0$ with an open set in $\mathbb{R}^n$, which we also denote $\mathcal{O}$. Pick $\zeta_0 \in \mathbb{R}^n \setminus 0$ such that $(z_0, \zeta_0) \in T^*\mathcal{O} \approx \mathcal{O} \times \mathbb{R}^n$ does not belong to the conormal bundle $\nu(\mathcal{F})$. The assertion is that if $\varphi \in C_0^\infty(\mathcal{O})$ is supported in a sufficiently small neighborhood of $z_0$ and if $\chi \in C^\infty(\mathbb{R}^n)$

is homogeneous of degree zero for $|\zeta| \geq 1$ and has support in a sufficiently small conic neighborhood of $\zeta_0$, then

$$\chi(D)(\varphi u) \in H^{k,p}(\mathbb{R}^n), \tag{1.8}$$

where $\chi(D)$ is the operation of multiplying the Fourier transform of its argument by $\chi(\zeta)$. Further material on this notion is given at the beginning of §2.

Theorem 1.1 follows from Theorem 1.2 because the hypothesis (1.6) is equivalent to

$$\bigcap_{j=1}^{N} \nu(\mathcal{F}_j) = 0.$$

Hence, under the hypotheses of Theorem 1.1, we have $\chi_j(D)(\varphi u) \in H^{k,p}$ for a collection $\chi_j$ that can be taken to sum to 1, yielding $u \in H^{k,p}$ locally in a neighborhood of each $z_0 \in M$.

As suggested in the previous two paragraphs, the proof of Theorem 1.2 involves an analysis of the behavior of the Fourier transform $\widehat{\varphi u}(\zeta)$, for $\zeta$ in a cone. This argument shares some common points with the proof of Theorem 2.6 in [HK], but there are significant differences, particularly in the use here of Calderon-Zygmund theory, which enables us to obtain a sharp result, without loss of derivatives. We will also give some extensions of Theorem 1.2 in §2, including a result in the endpoint case $p = \infty$ and results involving a fractional degree of smoothness along the leaves of $\mathcal{F}$, one of which has some of the flavor of the Hölder regularity result of [J].

The method of proof of Theorem 1.1 suggested in [dL], which had been carried out more fully on pp. 573–578 of [dLMM], involved the construction of an elliptic operator, of order $k = 2\ell$, with rough coefficients, of the following sort. Say $p \in M$ is contained in a neighborhood $\mathcal{O}$ and in turn $\overline{\mathcal{O}}$ is contained in $\cap_{j=1}^{N} \mathcal{O}_j$, on each of which we have $\Gamma_j : U_j \times V_j \to \mathcal{O}_j$, as in (1.1), yielding the foliation $\mathcal{F}_j$, satisfying hypotheses parallel to (1.2)–(1.4). We define the differential operator $P_j$ by

$$(P_j u) \circ \Gamma_j = \sum_{|\beta|=\ell} (-\partial_y)^\beta (\partial_y^\beta)(J_j \, u \circ \Gamma_j). \tag{1.9}$$

For each $j \in \{1, \dots, N\}$, $P_j$ is a differential operator on $\mathcal{O}$, of order $2\ell = k$. We identify $\mathcal{O}$ with an open subset of $\mathbb{R}^n$ and use Lebesgue measure on $\mathcal{O}$. The hypothesis (1.6) implies that $P = \sum_{j=1}^{N} P_j$ is an elliptic differential operator on $\mathcal{O}$, of order $k = 2\ell$. Furthermore, the hypothesis that each $\mathcal{F}_j$ is regular of order $(k, 0)$ implies that $P$ has continuous coefficients, i.e.,

$$P = \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha, \quad a_\alpha \in C^0(\mathcal{O}). \tag{1.10}$$

Now [dL] proposed to apply an elliptic regularity result to prove Theorem 1.1 (with $k = 2\ell$). Such a result is stated in Theorem 4 of [dL], accompanied by a reference to Chapter V of [ADN]. However, such a result is not to be found in [ADN], and indeed [dL]'s Theorem 4 as stated does not seem to be correct. Raphael de le Llave has pointed out to us that it is possible to adapt an argument presented on pp. 573–578 of [dLMM] to prove a special case of Theorem 1.3, in which one assumes $P = P^t$ and $P$ is strongly elliptic. Details will appear in a forthcoming note [dL2]. Theorem 1.3 below is the closest variant of [dL]'s Theorem 4 that seems to work. In this context, this result answers a question raised on p. 578 of [dLMM].

**Theorem 1.3.** *Let $P$ be an elliptic differential operator of order $k$ on an open set $\mathcal{O} \subset \mathbb{R}^n$. Assume*

$$P \quad \text{and} \quad P^t \quad \text{have continuous coefficients.} \tag{1.11}$$

*Then, given $p \in (0, \infty)$,*

$$u, \ Pu \in L^p_{\mathrm{loc}}(\mathcal{O}) \Longrightarrow u \in H^{k,p}_{\mathrm{loc}}(\mathcal{O}). \tag{1.12}$$

In Theorem 4 of [dL] it was only assumed that $P$ had continuous coefficients. The extra hypothesis in (1.11) serves to make sense of $Pu$, given the weak hypothesis that $u \in L^p_{\mathrm{loc}}(\mathcal{O})$. This result is also not contained in [ADN], and another goal here is to present a proof of Theorem 1.3. Note that if $P_j$ is given by (1.9), then (with $z = \Gamma(x, y)$)

$$\begin{aligned} \int u\,(\overline{P_j v})\,dz &= \sum_{|\beta|=\ell} \int \partial_y^\beta (J\,u \circ \Gamma)\ \partial_y^\beta (J\,\overline{v \circ \Gamma})\,dx\,dy \\ &= \int (P_j u)\,\overline{v}\,dz, \end{aligned} \tag{1.13}$$

and hence $P = \sum P_j$ satisfies

$$P = P^t, \tag{1.14}$$

so (1.11) holds in this setting.

Our treatment of Theorem 1.3 will be given in §3. We consider such an elliptic regularity result to have its own intrinsic interest, and in this spirit we establish such a result in a more general context, replacing (1.11) by:

$$P \quad \text{and} \quad P^t \quad \text{have coefficients in} \quad L^\infty \cap \mathrm{vmo}\,. \tag{1.15}$$

We also allow $P$ to be an elliptic $N \times N$ system. We discuss other variants in §4, in which the hypothesis (1.15) is strengthened and the result (1.12) is altered.

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## 2. Microlocal regularity

As Theorem 1.2 is a local result, we may assume $\mathcal{F}$ is a foliation, regular of order $(k,0)$, of an open set $\mathcal{O} \subset \mathbb{R}^n$, defined by $\Gamma$, as in (1.1), and that (1.2)–(1.4) hold. We assume

$$u \in H^{k,p}_{\mathcal{F}}(\mathcal{O}). \tag{2.1}$$

As indicated in the discussion right below the statement of Theorem 1.2, our goal is to establish the following. Pick any $z_0 \in \mathcal{O}$, and pick $\zeta_0 \in \mathbb{R}^n \setminus 0$ such that $(z_0, \zeta_0) \in T^*\mathcal{O} \approx \mathcal{O} \times \mathbb{R}^n$ does not belong to the conormal bundle $\nu(\mathcal{F})$. We claim that if $\varphi \in C_0^\infty(\mathcal{O})$ is supported in a sufficiently small neighborhood of $z_0$ and if $\chi \in C^\infty(\mathbb{R}^n)$ is homogeneous of degree zero for $|\zeta| \geq 1$ and is supported in a sufficiently small conic neighborhood of $\zeta_0$, then

$$\chi(D)(\varphi u) \in H^{k,p}(\mathbb{R}^n). \tag{2.2}$$

It is useful to mention the following equivalent characterization of the assertion that a function (or distribution) on the manifold $M$ is in $H^{k,p}$ microlocally on the complement of $\nu(\mathcal{F})$ in $T^*M \setminus 0$. Namely, this is also equivalent to the assertion that whenever $A$ is a classical pseudodifferential operator of order zero on $M$ (we write $A \in OPS^0(M)$), if the complete symbol of $A$ vanishes on a conic neighborhood of $\nu(\mathcal{F})$ in $T^*M \setminus 0$, then

$$Au \in H^{k,p}(M). \tag{2.3}$$

The equivalence of these two notions of being microlocally in $H^{k,p}$, given $p \in (1,\infty)$, is a consequence of basic properties of pseudodifferential operators acting on $L^p$-Sobolev spaces, as discussed, for example, in Chapter 13 of [T1]. It is in the latter characterization that $T^*M \setminus 0$ arises as the natural set on which to describe microlocal regularity.

To begin the analysis of $\chi(D)(\varphi u)$, note that

$$\widehat{\varphi u}(\zeta) = \int u(\Gamma(x,y))\varphi(\Gamma(x,y))J(x,y)e^{i\psi(x,y,\zeta)}\,dx\,dy, \tag{2.4}$$

where

$$\psi(x,y,\zeta) = -\Gamma(x,y)\cdot\zeta. \tag{2.5}$$

The hypothesis that $(z_0,\zeta_0)\notin\nu(\mathcal{F})$ implies that

$$d_y\psi(x_0,y_0,\zeta_0)\neq 0,\quad \text{where}\ \ \Gamma(x_0,y_0)=z_0. \tag{2.6}$$

Relabeling the $y$ coordinates, we can assume there is a neighborhood $\mathcal{U}$ of $z_0$ and a conic neighborhood $\mathcal{Z}$ of $\zeta_0$, and a $\delta>0$, such that

$$\Big|\frac{\partial\psi}{\partial y_1}(x,y,\zeta)\Big|\geq\delta|\zeta|,\quad \text{for}\ \ (\Gamma(x,y),\zeta)\in\mathcal{U}\times\mathcal{Z}. \tag{2.7}$$

Let

$$A(x,y,\zeta)=-\Big(i\frac{\partial\psi}{\partial y_1}(x,y,\zeta)\Big)^{-1}. \tag{2.8}$$

Use the identity

$$A(x,y,\zeta)\frac{\partial}{\partial y_1}e^{i\psi(x,y,\zeta)}=-e^{i\psi(x,y,\zeta)}, \tag{2.9}$$

and integrate by parts to deduce that

$$\widehat{\varphi u}(\zeta)=\int e^{i\psi(x,y,\zeta)}\Big(\frac{\partial}{\partial y_1}\circ A(x,y,\zeta)\Big)^k\big(u(\Gamma(x,y))\varphi(\Gamma(x,y))J(x,y)\big)\,dx\,dy, \tag{2.10}$$

for $\varphi\in C_0^\infty(\mathcal{U}),\ \zeta\in\mathcal{Z}$. Expand the differential operator

$$\Big(\frac{\partial}{\partial y_1}\circ A(x,y,\zeta)\Big)^k=\sum_{j=0}^k a_{kj}(x,y,\zeta)\frac{\partial^j}{\partial y_1^j}. \tag{2.11}$$

By (2.7), we have $a_{kj}(x,y,\zeta)$ continuous in $(x,y)\in\Gamma^{-1}(\mathcal{U})$, smooth in $\zeta\in\mathcal{Z}$, and homogeneous in $\zeta$ of degree $-k$.

Thus if we set

$$f_j(x,y)=\frac{1}{J(x,y)}\frac{\partial^j}{\partial y_1^j}\Big(u(\Gamma(x,y))\varphi(\Gamma(x,y))J(x,y)\Big),\quad 0\leq j\leq k, \tag{2.12}$$

we have

$$\widehat{\varphi u}(\zeta)=\sum_{j=0}^k\int e^{-iz\cdot\zeta}a_{kj}(\Gamma^{-1}(z),\zeta)f_j(\Gamma^{-1}(z))\,dz, \tag{2.13}$$

for $\zeta \in \mathcal{Z}$. Consequently, given $\chi \in C^\infty(\mathbb{R}^n)$, vanishing near $\zeta = 0$, homogeneous of degree zero for $|\zeta| \ge 1$, and supported in $\mathcal{Z}$, we have

$$\chi(D)(\varphi u) = \sum_{j=0}^{k} A_{kj}(f_j \circ \Gamma^{-1}), \tag{2.14}$$

where

$$A_{kj} f(z) = (2\pi)^{-n} \iint e^{i(z-z')\cdot\zeta} \chi(\zeta) a_{kj}(\Gamma^{-1}(z'), \zeta) f(z')\, dz'\, d\zeta. \tag{2.15}$$

To be more precise, we take

$$\chi(\zeta) = \chi_0(\zeta)(1 - \rho(\zeta)), \tag{2.16}$$

where $\chi_0 \in C^\infty(\mathbb{R}^n \setminus 0)$ is homogeneous of degree zero and supported in $\mathcal{Z}$, and $\rho \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 for $|\zeta| \le 1/2$ and to 0 for $|\zeta| \ge 1$.

The hypotheses on a foliation regular of order $(k, 0)$ clearly give

$$\begin{aligned} u \in H^{k,p}_{\mathcal{F}}(\mathcal{O}) &\Longrightarrow f_j \in L^p(U \times V) \\ &\Longrightarrow f_j \circ \Gamma^{-1} \in L^p(\mathcal{O}), \end{aligned} \tag{2.17}$$

for $0 \le j \le k$. Hence the desired conclusion (2.2) follows from the mapping property

$$\partial_z^\beta \circ A_{kj} : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad 0 \le j \le k, \quad |\beta| = k, \tag{2.18}$$

which we prove as follows. First note that applying $\partial_z^\beta$ to $A_{kj} f$ in (2.15) is effected by multiplying the integrand there by $(i\zeta)^\beta$. Hence we can write

$$\partial_z^\beta \circ A_{kj} = (I - \rho(D)) B_{kj},$$

use the elementary fact that $\rho(D) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, and note that the following result applies to $B_{kj}$.

**Proposition 2.1.** *Assume $b(z, \zeta)$ is homogeneous of degree zero in $\zeta$ and satisfies*

$$|\partial_\zeta^\gamma b(z, \zeta)| \le C_\gamma |\zeta|^{-|\gamma|}, \quad \forall\, \gamma \ge 0. \tag{2.19}$$

*Then the operator $B$, defined by*

$$Bf(z) = (2\pi)^{-n} \iint e^{i(z-z')\cdot\zeta} b(z', \zeta) f(z')\, dz'\, d\zeta \tag{2.20}$$

*has the mapping property*

$$B : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad \forall\, p \in (1,\infty). \tag{2.21}$$

This is a classical result of Calderon-Zygmund theory. We sketch the proof. The hypothesis (2.19) guarantees that

$$b(z',\zeta) = \sum_\mu b_\mu(z') Y_\mu\Bigl(\frac{\zeta}{|\zeta|}\Bigr), \tag{2.22}$$

where $\{Y_\mu\}$ is the orthonormal basis of $L^2(S^{n-1})$ consisting of spherical harmonics. The smoothness of $b(z,\zeta)$ in $\zeta$ yields the rapid decay

$$\|b_\mu\|_{L^\infty} \le C_N(1+|\mu|)^{-N}, \quad \forall\, N \in \mathbb{Z}^+, \tag{2.23}$$

while uniform estimates on the functions $Y_\mu$ and their derivatives, in concert with the Calderon-Zygmund estimates give, for

$$\mathcal{Y}_\mu f(z) = (2\pi)^{-n} \iint e^{i(z-z')\cdot\zeta} Y_\mu\Bigl(\frac{\zeta}{|\zeta|}\Bigr) f(z')\, dz'\, d\zeta, \tag{2.24}$$

estimates

$$\|\mathcal{Y}_\mu f\|_{L^p} \le C_p(1+|\mu|)^M \|f\|_{L^p}, \tag{2.25}$$

for $p \in (1,\infty)$, with finite $C_p$ and $M$, independent of $\mu$. Since

$$Bf = \sum_\mu \mathcal{Y}_\mu(b_\mu f), \tag{2.26}$$

the result (2.21) follows.

At this point, Theorem 1.2 is proven.

REMARK. Closely related to (2.17) is the identity

$$H^{0,p}_{\mathcal{F}}(M) = L^p(M),$$

itself a consequence of the absolute continuity hypothesis yielding (1.3), with continuous, positive $J(x,y)$. It is of interest to expand on this a bit.

Suppose $\mu$ is a (signed) measure on $M$, and suppose $M$ is covered by open sets $\mathcal{O}$ with the structures (1.1)–(1.4). For each such $\mathcal{O}$, we have a homeomorphism $\Sigma = \Gamma^{-1} : \mathcal{O} \to U \times V$, and $\mu$ pushes forward to define a measure $\Sigma_*\mu$ on $U \times V$. It follows from the hypotheses on $J$ that if $\Sigma_*\mu$ is absolutely continuous with respect to Lebesgue measure on $U \times V$ and defines a member of $L^p$, then $\mu \in L^p_{\text{loc}}(\mathcal{O})$. If this holds for all such $\mathcal{O}$, then $\mu \in L^p(M)$.

Having proven Theorem 1.2, we take a look at the endpoint case $p = \infty$. The characterization (1.5) still works to define $H^{k,\infty}_{\mathcal{F}}(M)$. The appropriate microlocal regularity to deduce for an element of this space involves the bmo-Sobolev space:

$$\mathfrak{h}^{k,\infty}(M) = \{u \in \text{bmo}(M) : \partial^\alpha u \in \text{bmo}(M),\ \forall\, |\alpha| \le k\}. \tag{2.27}$$

The following result complements Theorem 1.2.

**Proposition 2.2.** *In the setting of Theorem 1.2, if* $u \in H^{k,\infty}_{\mathcal{F}}(M)$*, then* $u$ *is in* $\mathfrak{h}^{k,\infty}$ *microlocally on any conic subset of* $T^*M \setminus 0$ *disjoint from* $\nu(\mathcal{F})$*.*

Here the conclusion is that $Au \in \mathfrak{h}^{k,\infty}(M)$ whenever $A \in OPS^0(M)$ has complete symbol that vanishes on a conic neighborhood of $\nu(\mathcal{F})$ in $T^*M \setminus 0$. The proof works as above; we need to complement the mapping property (2.18) with the result

$$\partial_z^\beta \circ A_{kj} : L^\infty(\mathbb{R}^n) \longrightarrow \text{bmo}(\mathbb{R}^n), \quad 0 \le j \le k, \quad |\beta| = k. \tag{2.28}$$

In turn, the proof of (2.18) given above works here, with the estimate (2.25) replaced by

$$\|\mathcal{Y}_\mu(1 - \rho(D))f\|_{\text{bmo}} \le C(1 + |\mu|)^M \|f\|_{L^\infty}. \tag{2.29}$$

This estimate follows from results given in [St], §IV.4.1.

From Proposition 2.2 we immediately get the following complement to Theorem 1.1. We state it explicitly here since it displays a sharp form of Theorem 2.6 of [HK].

**Corollary 2.3.** *In the setting of Theorem 1.1, we have*

$$u \in H^{k,\infty}_{\mathcal{F}_j}(M),\ \forall\, j \in \{1, \dots, N\} \Longrightarrow u \in \mathfrak{h}^{k,\infty}(M). \tag{2.30}$$

Next we extend Theorem 1.2 to cases involving fractional smoothness along the leaves of $\mathcal{F}$. To phrase our results, if we have $(x, y) \in \mathbb{R}^{n-m} \times \mathbb{R}^m = \mathbb{R}^n$, $v \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, we say

$$v \in H^{s,p}_y(\mathbb{R}^n) \Longleftrightarrow (1 - \Delta_y)^{s/2} v \in L^p(\mathbb{R}^n). \tag{2.31}$$

Assume $\mathcal{F}$ is a foliation regular of order $(k, 0)$ and $s \in [0, k]$. Then we say $u \in L^p(M)$ belongs to $H^{s,p}_{\mathcal{F}}(M)$ provided there is a cover of $M$ by open sets $\mathcal{O}$ as considered in (1.1)–(1.4) such that

$$\varphi\,(u \circ \Gamma) \in H^{s,p}_y(\mathbb{R}^n), \quad \forall\, \varphi \in C_0^\infty(U \times V). \tag{2.32}$$

It is clear that if $s = j$ is an integer in $[0, k]$ then $H^{s,p}_{\mathcal{F}}(M)$ coincides with $H^{j,p}_{\mathcal{F}}(M)$ as defined before. Also standard techniques in complex interpolation theory give

$$[L^p(M), H^{k,p}_{\mathcal{F}}(M)]_\theta = H^{k\theta,p}_{\mathcal{F}}(M), \quad 0 \le \theta \le 1. \tag{2.33}$$

Using this, we can establish the following.

**Proposition 2.4.** *In the setting of Theorem 1.2, if $u \in H^{s,p}_{\mathcal{F}}(M)$ with $s \in [0, k]$ and $p \in (1, \infty)$, then $u$ is microlocally in $H^{s,p}$ on any conic subset of $T^*M \setminus 0$ disjoint from $\nu(\mathcal{F})$.*

*Proof.* Take $A \in OPS^0(M)$, having complete symbol vanishing on a conic neighborhood of $\nu(\mathcal{F})$ in $T^*M \setminus 0$. We have shown that

$$A : H^{k,p}_{\mathcal{F}}(M) \longrightarrow H^{k,p}(M), \tag{2.34}$$

and of course

$$A : L^p(M) \longrightarrow L^p(M). \tag{2.35}$$

It follows from (2.33) and interpolation that

$$A : H^{s,p}_{\mathcal{F}}(M) \longrightarrow H^{s,p}(M), \quad \forall\, s \in [0, k], \tag{2.36}$$

given $p \in (1, \infty)$, and this proves the proposition.

Using Proposition 2.4, we can derive some results on Hölder regularity, as follows. Given a foliation $\mathcal{F}$, regular of order $(k, 0)$, given $\ell \in \mathbb{Z}^+$ with $0 \le \ell \le k - 1$ and $r \in (0, 1)$, and given $u \in C(M)$, we say $u \in C^{\ell,r}_{\mathcal{F}}(M)$ provided there is a cover of $M$ by open sets $\mathcal{O}$ as considered in (1.1)–(1.4) such that

$$|\partial_y^j u \circ \Gamma(x, y_1) - \partial_y^j u \circ \Gamma(x, y_2)| \le C|y_1 - y_2|^r, \quad 0 \le j \le \ell,\ (x, y_\nu) \in U \times V. \tag{2.37}$$

It is elementary that

$$C^{\ell,r}_{\mathcal{F}}(M) \subset H^{\ell+s,p}_{\mathcal{F}}(M), \quad \forall\ s < r,\ p < \infty. \tag{2.38}$$

Hence $u \in C^{\ell,r}_{\mathcal{F}}(M) \Rightarrow Au \in H^{\ell+s,p}(M)$, $\forall\, p < \infty$, when $A$ is as in (2.36). Furthermore,

$$\bigcap_{p<\infty} H^{\ell+s,p}(M) \subset C^{\ell+\sigma}(M), \quad \forall\ \sigma < s, \tag{2.39}$$

so we have

$$A : C^{\ell,r}_{\mathcal{F}}(M) \longrightarrow C^{\ell+\sigma}(M), \quad \forall\ \sigma < r, \tag{2.40}$$

when $A \in OPS^0(M)$ has complete symbol vanishing in a conic neighborhood of $\nu(\mathcal{F})$ in $T^*M \setminus 0$. This immediately leads to the following result.

**Proposition 2.5.** *In the setting of Theorem 1.1, given* $0 \leq \ell \leq k-1$, $0 < r < 1$, *we have*

$$u \in C^{\ell,r}_{\mathcal{F}_j}(M),\ \forall\, j \in \{1,\dots,N\} \Longrightarrow u \in C^{\ell+\sigma}(M), \quad \forall\, \sigma < r. \tag{2.41}$$

*(Only the case* $\ell \geq 1$ *is nontrivial. When* $\ell = 0$ *one can clearly take* $\sigma = r$*.)*

This can be compared with the result of [J]. The result of [J] is better in two respects. First, one can take $\sigma = r$ in the conclusion. Second, and of great interest, is that [J] does not need the absolute continuity hypothesis (1.3)–(1.4). On the other hand, [J] works with only two foliations, while Proposition 2.5 works with any number $N$ of foliations.

## 3. Elliptic regularity

We prove Theorem 1.3, with the hypothesis (1.11) weakened to (1.15). The proof involves two steps. The first step is a globalization argument, in which we produce an elliptic operator $L$ on the torus $\mathbb{T}^n$ which is shown to be invertible from $H^{s+k,p}(\mathbb{T}^n)$ to $H^{s,p}(\mathbb{T}^n)$, for $-k \leq s \leq 0$. The second step is a localization argument, obtaining results of the form (1.12) from the corresponding global results for $L$, via cutoffs and commutator estimates. We make use of results on pseudodifferential operators with rough coefficients, treated in [T2].

In order to obtain such $L$, we make some preliminary transformations of $P$. Clearly the local regularity result (1.12) holds for $P$ if it holds for the $2N \times 2N$ system

$$P_2 = \begin{pmatrix} 0 & P^t \\ P & 0 \end{pmatrix}. \tag{3.1}$$

We begin to construct an operator on the torus. Without loss of generality, we can assume $\mathcal{O} \subset \mathbb{R}^n$ is bounded. Put this set in a box and identify opposite edges, to regard $\mathcal{O} \subset \mathbb{T}^n$. If we want to establish local regularity on a neighborhood $\Omega$ of $x_0 \in \mathcal{O}$, pick $\varphi \in C_0^\infty(\mathcal{O})$, equal to 1 on $\overline{\Omega}$, satisfying $0 \leq \varphi \leq 1$, set $\psi = 1 - \varphi$, and form

$$L_2 u = \varphi P_2(\varphi u) + i\psi(-\Delta)^{k/2}(\psi u), \tag{3.2}$$

where $u$ takes values in $\mathbb{C}^{2N}$ and $(-\Delta)^{k/2}$ acts on $\psi u$ componentwise. This is a differential operator if $k$ is even, and otherwise not, but in any case

$$u \in L^p(\mathbb{T}^n) \Longrightarrow L_2 u = P_2 u \quad \text{on} \quad \Omega. \tag{3.3}$$

Hence, to obtain the desired local regularity (1.12), on $\Omega$, it suffices to establish

$$u,\ L_2 u \in L^p_{\mathrm{loc}}(\Omega) \Longrightarrow u \in H^{k,p}_{\mathrm{loc}}(\Omega). \tag{3.4}$$

We note that $L_2$ is a pseudodifferential operator with rough symbol; more precisely,

$$L_2 \in OP(L^\infty \cap \mathrm{vmo}) S^k_{\mathrm{cl}}(\mathbb{T}^n). \tag{3.5}$$

The symbol class $(L^\infty \cap \mathrm{vmo}) S^k_{\mathrm{cl}}$ is as described in §11, Chapter I of [T2]. Furthermore, since $P_2 = P_2^t$ and $(-\Delta)^{k/2}$ is scalar, it is readily established that $L_2$ is elliptic. In addition, it is important (and easily seen) that we have

$$L_2^t \in OP(L^\infty \cap \mathrm{vmo}) S^k_{\mathrm{cl}}(\mathbb{T}^n). \tag{3.6}$$

We again desire formal self-adjointness, so we form the $4N \times 4N$ system

$$L_1 = \begin{pmatrix} 0 & L_2^t \\ L_2 & 0 \end{pmatrix}, \tag{3.7}$$

and note that it suffices to establish the result (3.4) with $L_2$ replaced by $L_1$, which is also elliptic in $OP(L^\infty \cap \mathrm{vmo}) S^k_{\mathrm{cl}}(\mathbb{T}^n)$. We are ready for some global results, starting with:

**Lemma 3.1.** *For each $p \in (1,\infty)$, the map*

$$L_1 : H^{k,p}(\mathbb{T}^n, V) \longrightarrow L^p(\mathbb{T}^n, V) \tag{3.8}$$

*is Fredholm, of index zero.*

Here $H^{k,p}(\mathbb{T}^n, V)$ denotes the space of functions on $\mathbb{T}^n$ with values in $V = \mathbb{C}^{4N}$, having the indicated regularity.

*Proof.* The Fredholm property is equivalent to the statement that

$$A = L_1(1-\Delta)^{-k/2} \in OP(L^\infty \cap \mathrm{vmo}) S^0_{\mathrm{cl}} \tag{3.9}$$

is Fredholm on $L^p(\mathbb{T}^n, V)$. Results of §11, Chapter I of [T2] lead to the construction of an operator $B \in OP(L^\infty \cap \mathrm{vmo}) S^0_{\mathrm{cl}}$ which is a Fredholm inverse of $A$ on $L^p(\mathbb{T}^n, V)$ for each $p \in (0,\infty)$. The proof that the index is zero then follows from the standard homotopy argument involving

$$A_\tau = \tau A + (1-\tau) i I, \quad 0 \le \tau \le 1, \tag{3.10}$$

which, we note, is elliptic in $OP(L^\infty \cap \text{vmo})S^0_{\text{cl}}$ for each $\tau \in [0,1]$, since the principal symbol of $A$ is self-adjoint and elliptic and the principal symbol of the identity $I$ is scalar and elliptic.

Remark. Behind the construction of the Fredholm inverse $B$ mentioned above is a fairly deep commutator estimate of [CRW]:

$$\|Q(fu) - fQu\|_{L^p} \le C\|f\|_{\text{bmo}}\|u\|_{L^p},$$

given a zero-order pseudodifferential operator $Q$. In the case where one uses the hypothesis (1.11), rather than the weaker hypothesis (1.15), this step is correspondingly more elementary.

We next give an invertibility result.

**Lemma 3.2.** *There is a discrete set $\mathcal{C} \subset \mathbb{R}$ such that for all $\lambda \in \mathbb{C} \setminus \mathcal{C}$ we have isomorphisms*

$$L_1 - \lambda I : H^{k,p}(\mathbb{T}^n, V) \xrightarrow{\approx} L^p(\mathbb{T}^n, V), \tag{3.11}$$

*for all $p \in (1, \infty)$.*

*Proof.* In view of Lemma 3.1, isomorphism holds in (3.11) if and only if the null space of $L_1 - \lambda I$ in $H^{k,p}(\mathbb{T}^n, V)$ is zero. We first establish this for $p = 2$, using the following elementary Hilbert space result.

**Lemma 3.3.** *Let $H_0$ be a Hilbert space, $H_1$ a dense linear subspace with a Hilbert space structure such that the injection $H_1 \hookrightarrow H_0$ is compact. Let $T$ be a symmetric operator on $H_0$ with domain $\mathcal{D}(T) = H_1$, and assume*

$$T : H_1 \longrightarrow H_0 \tag{3.12}$$

*is Fredholm, of index zero. Then we have*

$$T - \lambda I : H_1 \xrightarrow{\approx} H_0 \tag{3.13}$$

*for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Hence $T$ is self-adjoint (with domain $\mathcal{D}(T) = H_1$); $T$ has compact resolvent, and (3.13) holds except for $\lambda$ in a discrete set $\mathcal{C} \subset \mathbb{R}$.*

*Proof.* Under our hypotheses, we see that (3.13) is an isomorphism if and only if the null space of $T - \lambda I$ in $H_1$ is zero. Given $u \in H_1$ with $(T - \lambda I)u = 0$, we have

$$\lambda \|u\|^2_{H_0} = (Tu, u) = (u, Tu) = \overline{\lambda}\|u\|^2_{H_0},$$

so (3.13) holds whenever $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The rest of the assertions of the lemma are routine consequences.

Applying Lemma 3.3 to $H_0 = L^2(\mathbb{T}^n, V)$, $H_1 = H^{k,2}(\mathbb{T}^n, V)$, $T = L_1$, we see that (3.11) holds for $p = 2$, $\lambda \in \mathbb{R} \setminus \mathcal{C}$, where $\mathcal{C}$ is a discrete subset of $\mathbb{R}$. At this point we have *a fortiori* that the null space of $L_1 - \lambda I$ in $H^{k,p}(\mathbb{T}^n, V)$ is zero whenever $p > 2$, so (3.11) holds for $2 \le p < \infty$. To proceed, note that when $\lambda \in \mathbb{R}$ and $L_1^t = L_1$, (3.11) is equivalent to

$$L_1 - \lambda I : L^q(\mathbb{T}^n, V) \xrightarrow{\approx} H^{-k,q}(\mathbb{T}^n, V), \quad q = p'. \tag{3.14}$$

In particular we see that the null space of $L_1 - \lambda I$ in $L^q(\mathbb{T}^n, V)$ is zero for $1 < q \le 2$. This implies the null space of $L_1 - \lambda I$ in $H^{k,p}(\mathbb{T}^n, V)$ is zero for $1 < p \le 2$, so we have (3.11) for each $p \in (1, \infty)$, and Lemma 3.2 is proven.

Note that we also have (3.14) for all $q \in (1, \infty)$, whenever $\lambda \in \mathbb{R} \setminus \mathcal{C}$. At this point, fix $\lambda_1 \in \mathbb{R} \setminus \mathcal{C}$ and set

$$L = L_1 - \lambda_1 I. \tag{3.15}$$

We have

$$L : H^{k,p}(\mathbb{T}^n, V) \xrightarrow{\approx} L^p(\mathbb{T}^n, V), \quad L : L^p(\mathbb{T}^n, V) \xrightarrow{\approx} H^{-k,p}(\mathbb{T}^n, V), \tag{3.16}$$

whenever $p \in (1, \infty)$. Consequently we have a two-sided inverse

$$E = L^{-1}, \tag{3.17}$$

satisfying

$$\begin{aligned} E : L^p(\mathbb{T}^n, V) &\longrightarrow H^{k,p}(\mathbb{T}^n, V), \\ E : H^{-k,p}(\mathbb{T}^n, V) &\longrightarrow L^p(\mathbb{T}^n, V), \end{aligned} \tag{3.18}$$

for $1 < p < \infty$. To be pedantic, we might label these maps $E_{0p}$ and $E_{1p}$, respectively, but in this case it is readily verified that these various maps coincide on the intersections of their respective domains, so there is no harm in giving them a common label. Furthermore, by interpolation we have

$$E : H^{s,p}(\mathbb{T}^n, V) \longrightarrow H^{s+k,p}(\mathbb{T}^n, V), \quad 1 < p < \infty, \ -k \le s \le 0. \tag{3.19}$$

With these global results in hand, we are ready to localize. As noted, it remains to prove (3.4) with $L_2$ replaced by $L_1$, which in turn can clearly be replaced by $L$. Thus the remaining task is to prove the following.

**Proposition 3.4.** *Given $\Omega \subset \mathbb{T}^n$ as in (3.4), $1 < p < \infty$,*

$$u, Lu \in L^p_{\mathrm{loc}}(\Omega) \Longrightarrow u \in H^{k,p}_{\mathrm{loc}}(\Omega). \tag{3.20}$$

The key to obtaining the local regularity result (3.20) is to combine (3.16)–(3.19) with the following commutator estimate.

**Lemma 3.5.** *Let $\mathcal{U} \subset \overline{\mathcal{U}} \subset \Omega$ and assume $\partial\mathcal{U}$ is smooth. Then, given $\rho \in C_0^\infty(\mathcal{U})$, we have, for $1 < p < \infty$,*

$$[L, \rho] : H^{\sigma,p}(\mathcal{U}) \longrightarrow H^{\sigma-(k-1),p}(\mathbb{T}^n), \quad 0 \le \sigma \le k-1. \tag{3.21}$$

*Proof.* From (3.2) we have that $L$ coincides with a differential operator on $\Omega$. Hence

$$[L, \rho]u = \sum_{|\alpha| \le k} b_\alpha(x)[\partial^\alpha, \rho]u, \quad b_\alpha \in L^\infty, \tag{3.22}$$

which gives (3.21) for $\sigma = k-1$. Since $L = L^t$, we also have

$$[L, \rho]u = \sum_{|\alpha| \le k} [(-\partial)^\alpha, \rho](b_\alpha^* u), \tag{3.23}$$

which gives (3.21) for $\sigma = 0$. The rest of (3.21) follows by interpolation.

To begin the proof of Proposition 3.4, assume $u, Lu \in L^p_{\mathrm{loc}}(\Omega)$, and pick $\rho \in C_0^\infty(\Omega)$. Pick $\mathcal{U} \subset \overline{\mathcal{U}} \subset \Omega$ such that $\rho \in C_0^\infty(\mathcal{U})$. We have

$$L(\rho u) = \rho L u + [L, \rho]u, \tag{3.24}$$

with

$$\rho L u \in L^p(\mathbb{T}^n), \quad [L, \rho]u \in H^{-k+1,p}(\mathbb{T}^n), \tag{3.25}$$

and hence

$$\rho u = E(\rho L u + [L, \rho]u) \in H^{1,p}(\mathbb{T}^n). \tag{3.26}$$

This allows us to replace the hypothesis in (3.20) by

$$u \in H^{1,p}_{\mathrm{loc}}(\Omega), \quad Lu \in L^p_{\mathrm{loc}}(\Omega). \tag{3.27}$$

Now if $k \ge 2$, we see that (3.24) holds with

$$\rho L u \in L^p(\mathbb{T}^n), \quad [L, \rho]u \in H^{-k+2,p}(\mathbb{T}^n), \tag{3.28}$$

and hence

$$\rho u = E(\rho L u + [L, \rho]u) \in H^{2,p}(\mathbb{T}^n). \tag{3.29}$$

Thus we can say $u \in H^{2,p}_{\mathrm{loc}}(\Omega)$. Iterating this argument (if $k > 2$) we arrive at the desired result (3.20).

Since the result we have established is stronger than Theorem 1.3, we close this section with a formal statement of our result, including one further small refinement.

**Theorem 3.6.** *Let $P$ be an elliptic differential operator of order $k$ on an open set $\mathcal{O} \subset \mathbb{R}^n$ (perhaps an $N \times N$ system). Assume*

$$P \ \text{ and } \ P^t \ \text{ have coefficients in } \ L^\infty \cap \mathrm{vmo}. \tag{3.30}$$

*Let $1 < q < p < \infty$. Then*

$$u \in L^q_{\mathrm{loc}}(\mathcal{O}), \ Pu \in L^p_{\mathrm{loc}}(\mathcal{O}) \Longrightarrow u \in H^{k,p}_{\mathrm{loc}}(\mathcal{O}). \tag{3.31}$$

The only extra wrinkle here is that the hypothesis $u \in L^p_{\mathrm{loc}}$ has been weakened to $u \in L^q_{\mathrm{loc}}$. But the work already done then yields $u \in H^{k,q}_{\mathrm{loc}}(\mathcal{O})$, hence $u \in L^{q_1}_{\mathrm{loc}}(\mathcal{O})$ for some $q_1 > q$. Iterating this argument a finite number of times produces the conclusion (3.31).

In the next section we discuss relaxing further the hypothesis on $u$ in (3.31).

## 4. Elliptic regularity with measures $u$ or with $Pu$ in bmo

Here we obtain some variants of Theorem 3.6. In the first result, we relax the hypothesis in (3.31) that $u \in L^q_{\mathrm{loc}}(\mathcal{O})$ for some $q > 1$, to the hypothesis that $u$ is a measure on $\mathcal{O}$, i.e., $u \in \mathcal{M}(\mathcal{O})$. In return, we require a little more regularity on the coefficients of $P^t$.

**Proposition 4.1.** *In the setting of Theorem 3.6, add to (3.30) the hypothesis*

$$P^t \ \text{ has Hölder continuous coefficients.} \tag{4.1}$$

*Then, given $p \in (1, \infty)$,*

$$u \in \mathcal{M}(\mathcal{O}), \ Pu \in L^p_{\mathrm{loc}}(\mathcal{O}) \Longrightarrow u \in H^{k,p}_{\mathrm{loc}}(\mathcal{O}). \tag{4.2}$$

*Proof.* The hypothesis (4.1) implies that $P$ can be written as follows:

$$Pu = \sum_{|\alpha| \le k} \partial^\alpha (a_\alpha(x) u), \quad a_\alpha \in C^r, \tag{4.3}$$

for some $r \in (0,1)$. We next use *symbol smoothing*, as described in Proposition 9.9, Chapter 13, of [T1], to write

$$\begin{aligned} P &= P^\# + P^b, \qquad P^\# \in OPS^k_{1,\delta}, \ \text{elliptic}, \\ P^b &= \sum_{|\alpha| \le k} \partial^\alpha A^b_\alpha, \qquad A^b_\alpha \in OPC^r S^{-r\delta}_{1,\delta}, \end{aligned} \tag{4.4}$$

where $\delta \in (0,1)$ is selected. The operator $P^\#$ has a parametrix $E^\# \in OPS^{-k}_{1,\delta}$, and we have

$$u = E^\#(Pu) - E^\#(P^b u), \quad \bmod C^\infty. \tag{4.5}$$

Now $Pu \in L^p_{\mathrm{loc}} \Rightarrow E^\#(Pu) \in H^{k,p}_{\mathrm{loc}}$, while

$$\begin{aligned} u \in \mathcal{M}(\mathcal{O}) &\Longrightarrow A^b_\alpha u \in L^q_{\mathrm{loc}}(\mathcal{O}) \text{ (for some } q > 1) \\ &\Longrightarrow E^\# P^b u \in L^q_{\mathrm{loc}}(\mathcal{O}). \end{aligned} \tag{4.6}$$

Hence the right side of (4.5) belongs to $L^q_{\mathrm{loc}}(\mathcal{O})$ for some $q > 1$. We are now in a setting where Theorem 3.6 applies, to yield the implication (4.2) and complete the proof of Proposition 4.1.

Here is another variant of Theorem 3.6. Recall that $\mathfrak{h}^{k,\infty}$ denotes a bmo-Sobolev space; cf. (2.27).

**Proposition 4.2.** *In the setting of Theorem 3.6, add the hypothesis*

$$P \ \text{ has Hölder continuous coefficients.} \tag{4.7}$$

*Then, given* $p \in (1,\infty)$,

$$u \in L^p_{\mathrm{loc}}(\mathcal{O}),\ Pu \in \mathrm{bmo}(\mathcal{O}) \Longrightarrow u \in \mathfrak{h}^{k,\infty}_{\mathrm{loc}}(\mathcal{O}). \tag{4.8}$$

*Proof.* By Theorem 3.6 we have $u \in H^{k,q}_{\mathrm{loc}}(\mathcal{O})$ for all $q < \infty$, so it remains to obtain

$$u \in H^{k,q}_{\mathrm{loc}}(\mathcal{O}),\ Pu \in \mathrm{bmo}(\mathcal{O}) \Longrightarrow u \in \mathfrak{h}^{k,\infty}_{\mathrm{loc}}(\mathcal{O}), \tag{4.9}$$

provided $P$ is elliptic of order $k$ and (4.7) holds.

This time we have $Pu = \sum_{|\alpha|\le k} a_\alpha(x)\partial^\alpha u$ with $a_\alpha \in C^r$ for some $r \in (0,1)$. We use the following result of symbol smoothing:

$$\begin{aligned} P = P^\# + P^b, &\qquad P^\# \in OPS^k_{1,\delta}, \text{ elliptic}, \\ P^b = \sum_{|\alpha|\le k} A^b_\alpha \partial^\alpha, &\qquad A^b_\alpha \in OPC^r S^{-r\delta}_{1,\delta}, \end{aligned} \tag{4.10}$$

with $\delta \in (0,1)$ selected. As above we have a parametrix $E^\# \in OPS^{-k}_{1,\delta}$ for $P^\#$, and the identity (4.5) holds. Now $Pu \in \mathrm{bmo}(\mathcal{O}) \Rightarrow E^\#(Pu) \in \mathfrak{h}^{k,\infty}_{\mathrm{loc}}$, while, given $q$ sufficiently large,

$$\begin{aligned} u \in H^{k,q}_{\mathrm{loc}}(\mathcal{O}) &\Longrightarrow A^b_\alpha \partial^\alpha u \in L^\infty_{\mathrm{loc}}(\mathcal{O}) \\ &\Longrightarrow E^\# P^b u \in \mathfrak{h}^{k,\infty}_{\mathrm{loc}}(\mathcal{O}). \end{aligned} \tag{4.11}$$

Hence the right side of (4.5) belongs to $\mathfrak{h}^{k,\infty}_{\text{loc}}(\mathcal{O})$, and Proposition 4.2 is proven.

Remark 1. With further effort, it seems, one can relax the Hölder hypothesis in (4.1) and (4.7) to some Dini-type condition on the coefficients of $P^t$ (or $P$), but the analysis gets fairly technical and we do not present it here.

Remark 2. To be sure, one can relax the regularity hypotheses on the coefficients of the *lower order* derivatives in $P$ and $P^t$, in Theorem 3.6 and in Propositions 4.1 and 4.2, using familiar techniques. We leave consideration of this matter to the interested reader.

## References


[ADN] S. Agmon, A. Douglis, and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, Comm. Pure Appl. Math. 12 (1959), 623–727.

[CRW] R. Coifman, R. Rochberg, and G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math. 103 (1976), 611–635.

[dL] R. de la Llave, Remarks on Sobolev regularity in Anosov systems, Ergod. Th. and Dyn. Sys. 21 (2001), 1139–1180.

[dL2] R. de la Llave, Personal communication.

[dLMM] R. de la Llave, J. Marco, and R. Moriyon, Canonical perturbation theory of Anosov systems and regularity results for the Livsic cohomology equation, Annals of Math. 123 (1986), 537–611.

[HK] S. Hurder and A. Katok, Differentiability, rigidity, and Godbillon-Vey classes for Anosov flows, Publ. Math. de l'IHES 72 (1990), 5–61.

[J] J.-L. Journé, A regularity lemma for functions of several variables, Revista Mat. Iberoamericana 42 (1988), 187–193.

[St] E. Stein, Harmonic Analysis, Princeton Univ. Press, Princeton, N.J., 1993.

[T1] M. Taylor, Partial Differential Equations, Vol. 3, Springer-Verlag, New York, 1996.

[T2] M. Taylor, Tools for PDE, Math. Surv. and Monogr., Vol. 81, AMS, Providence, R.I., 2000.



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