Intuitive and Counterintuitive Energy Flux

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Abstract. This theme of this paper is the computation and estimation of the flux and absolute flux of energy across hypersurfaces for evolution equations. The main surprise is that for smooth rapidly decreasing solutions of the wave equation or Shrödinger equation in dimensions greater than one, the absolute flux of energy across a hyperplane can be arbitrarily large compared to the total energy. For the heat equation the absolute flux never exceeds half the energy.

§0. Introduction.

The starting points are the basic differential energy conservation laws. Three model cases are; D'Alembert's wave equation for $u(t,x)$

$$\Box u = 0\,, \qquad \Box := \frac{\partial^2}{\partial t^2} - \Delta := \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}\,, \tag{1}$$

with conservation law

$$0 = \mathrm{Re}\,\big(\overline{u}_t \Box u\big) = \partial_t\Big(\frac{|u_t|^2 + |\nabla_x u|^2}{2}\Big) - \sum_{j=1}^{d} \partial_j \Big(\mathrm{Re}\,\big(\overline{u}_t \partial_j u\big)\Big)\,, \tag{2}$$

the heat equation

$$u_t = \Delta\, u\,, \tag{3}$$

with conservation law

$$0 = \partial_t(u) - \sum_{j=1} \frac{\partial}{\partial x_j}\Big(\frac{\partial u}{\partial x_j}\Big)\,, \tag{4}$$

and, Schrödinger's equation

$$u_t = i\,\Delta\, u\,, \tag{5}$$

with conservation law

$$0 = 2\mathrm{Re}\,\Big(\overline{u}(u_t - i\,\Delta u)\Big) = \partial_t |u|^2 + \sum_j \partial_j \big(2\,\mathrm{Im}\,(\overline{u}\,\partial_j u)\big)\,. \tag{6}$$

Each has the form

$$\partial_t e + \sum_j \partial_j f_j = 0\,. \tag{7}$$

Integrating the conservation law over $[0,t]\times\Omega$ for a nice subset $\Omega\subset\mathbb{R}^d$ yields

$$\int_\Omega e(t,x)\ dx - \int_\Omega e(0,x)\ dx + \int_{[0,t]\times\partial\Omega} \sum_j f_j \nu_j\ dt\, d\sigma = 0\,. \tag{8}$$

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Here ν is the unit outward normal to Ω and $d\sigma$ is the surface area element of $\partial\Omega$.

The interpretation of the last identity is that $\nu.f\ d\sigma$ gives the rate of flow of energy out of Ω through the element of surface $d\sigma$. Thus the vectorial quantity f gives the flux of energy. A rarely made but interesting observation is that the ratio f/e defines a velocity which for conservative systems like the wave equation and Schrödinger's equation gives the group velocity of plane waves.

For regular solutions sufficiently rapidly decaying as $|x| \to \infty$, one can take $\Omega = \mathbb{R}^d$ and conclude conservation of the total energy

$$E(u) \ := \ \int_{\mathbb{R}^d} e(t,x)\ dx\,. \tag{9}$$

The conservation laws bound the change in energy in terms of the size of f. We investigate the converse direction. Given a bound on the energy E can one deduce a bound on the size of f? Can one bound the flux in terms of the total energy? An intuitively obvious bound asserts that the absolute flux of energy across a hyperplane should not exceed the total energy. This is true the heat equation and for the wave equation in dimension 1. We prove that the result is radically wrong for the wave and Schrdöinger equations in dimensions $d > 1$. For conservative multidimensional systems, falsehood is the usual result.

Consider first the positive result for the heat equation. Gehring [G1] showed that the integral from 0 to ∞ of the absolute value of the heat flux across $x = 0$ for the one dimensional heat equation is never larger than one half the total energy. In §1 we extend this result to higher dimensions and identify the cases of equality answering questions raised by Gehring. I would like to take this opportunity to thank him for this inspiration.

The negative results in this paper are more interesting. For the wave equation in $\mathbb{R}^d$ with $d \geq 2$, consider the flux of energy from left to right across the plane $\{x_1 = 0\}$ for $a < t < b$

$$\int_{[a,b]\times\{x_1=0\}} -\mathrm{Re}\left(\overline{u}_t\ \frac{\partial u}{\partial x_1}\right)\ dt\,dx_2\ \ldots\ dx_d\,.$$

The basic energy identity (8) implies that this quantity is no larger than the total energy. If you think that propagation for the wave equation tends to be rectilinear, all the energy will cross $\{x_1 = 0\}$ at some time. That which crosses from left to right will count positive while the energy crossing in the opposite direction counts negative. If one takes the absolute value of the flux one will always count positive. This suggests that

$$\int_{]-\infty,\infty[\times\{x_1=0\}} \left|\mathrm{Re}\left(\overline{u}_t\ \frac{\partial u}{\partial x_1}\right)\right|\ dt\,dx_2\ \ldots\ dx_d \ = \ E$$

This is not correct since energy arriving from the left and right at the same time can have cancelling flux suggesting the refined guess that

$$\int_{]-\infty,\infty[\times\{x_1=0\}} \left|\mathrm{Re}\left(\overline{u}_t\ \frac{\partial u}{\partial x_1}\right)\right|\ dt\,dx_2\ \ldots\ dx_d \ \leq \ E \tag{10}$$

You can imagine Maxwell's demons distributed along $\{x_1 = 0\}$ counting the energy (or photons) which flow by and counting all flow as positive. The inequality asserts the natural expectation that the demon will not count more than the total energy.

This intuitive estimate is correct when the dimension is equal to one and is radically wrong when the dimension is greater than one. In §2, we show that for any small $\mu > 0$ and any large N there is a solution the wave equation with C_0^∞ Cauchy data and satisfying

$$\int_{[0,\mu]\times\{x_1=0\}} \left| \mathrm{Re}\left(\overline{u}_t\, \frac{\partial u}{\partial x_1}\right) \right| \,dt\, dx_2\, \ldots\, dx_d \;\geq\; N\,E \tag{11}$$

The absolute value of the flux across the plane $\{x_1 = 0\}$ is as large as one likes compared to the total energy.

Analogous negative results are valid for Shrödinger's equation (presented in §3) and Maxwell's equations.

The examples exhibiting the behavior (11) are sums of two asymptotic solutions of WKB type. The phases are linear, so this is not a focusing effect. It is a result of interaction. It would be natural to suspect that a single solution moving parallel to the boundary would be a good candidate. It is not. The constructions are related but quite different in the cases of the wave and Schrödinger equations.

In addition to the negative results, we present several positive results about the flux which follow intuitions somewhat like those leading to the incorrect guess (11). For example, the total flux through the boundary of a nice bounded set Ω vanishes,

$$\int_{]-\infty,\infty[\times\partial\Omega} \mathrm{Re}\left(\overline{u}_t\, \frac{\partial u}{\partial x_1}\right) \,dt\, d\sigma \;=\; 0\,.$$

This identity corresponds to the intuition that for all but a negligible set of rays, the number of enties into Ω is equal to the number of exits. Another such identity is an identity asserting that the total flux across the boundary of a nice open cone is equal to the energy in modes with group velocity pointing into the cone minus the energy in modes whose group velocity points out. This is a wave equation analogue of results of Dollard [D] for the Schrödinger equation.

The paper is organized as follows. The short first section discusses the heat equation. The second section gives a detailed account for the wave equation. A brief final section describes the changes needed for Schrödinger's equation.

§1. The heat equation.

The heat equation

$$u_t \;=\; \Delta\, u\,, \qquad \Delta := \sum_{j=1}^{d} \frac{\partial^2}{\partial^2 x_j}\,, \qquad u = u(t,x)\,, \qquad x \;=\; (x_1, x_2, \ldots, x_d) \;:= (x_1, x')\,,$$

describes the dynamics of the temperature $u(t,x)$ as a function on $\overline{\mathbb{R}_+} \times \mathbb{R}^d$.

The equation models a medium with constant heat capacity, so the temperature is proportional to the energy density per unit volume. The differential form of the law of conservation of energy is (0.4). Integrating dx shows that the total energy

$$E \;=\; E(u) \;:=\; \int u(t,x)\; dx$$

is independent of $t \geq 0$. The energy current is $-\nabla_x u$, and, with ν equal to the unit outward normal to Ω,

$$\int_{[a,b]\times\partial\Omega} -\nu.\nabla u \;\, dt\, d\sigma$$

is the total flux of energy out of Ω during the time interval $a \le t \le b$.

When $u > 0$, the energy flux can be written as

$$u \, \frac{-\nabla_x u}{u} \;=\; u \, \Big(- \nabla_x \ln u \Big)\,.$$

Since u is the energy density, one identifies $\nabla_x \ln u$ as a velocity of transport. This idea is the basis of a particle method for the numerical solution of the heat equation (see [L-M]).

In [G1, Theorem 16], F. Gehring proves that when $d = 1$ nonnegative solutions $u(t, x)$ of the heat equation $u_t = u_{xx}$ in $t > 0$ satisfy

$$\int_0^\infty |u_x(0,t)| \; dt \;\le\; \frac{1}{2} \int_{\mathbb{R}} u(0,x) \; dx\,.$$

The estimate shows that at most one half of the energy flows across $\{x_1 = 0\}$. He was motivated by an estimate of Féjer and Riesz in the theory of conformal mappings. In this section we

i. Show that this bound on the flux extends to higher dimensions with the same constant $1/2$.

ii. Complement the inequality with an identity which shows that the estimate is sharp and identifies the cases of equality.

We present the results for tempered solutions of the heat equation with initial data which are finite Borel measures rather than the class of nonnegative solutions. The case of nonnegative solutions follows from that and vice versa.

The formulas of the next result involve

$$\mathrm{sgn}\,(s) \;:=\; \begin{cases} 1 & \text{if} \quad s > 0 \\ -1 & \text{if} \quad s < 0 \\ 0 & \text{if} \quad s = 0. \end{cases}$$

The Schwartz space of rapidly decreasing smooth functions and it dual of tempered distributions are denoted as usual $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ respectively.

Theorem 1.1. *Suppose that $u(t,x) \in C\big([0,\infty[\,;\, \mathcal{S}'(\mathbb{R}^d)\big)$ satisfies the heat equation $u_t = \Delta u$ in $t > 0$ and has initial value, $u(0,x) = d\mu$ which is a finite Borel measure. Then the flux $\partial u/\partial x_1$ is absolutely integrable on $[0,\infty[\times\{x_1 = 0\}$ and the total absolute flux is no larger than half the energy,*

$$\int_{[0,\infty[\times\{x_1=0\}} \Big|\frac{\partial u}{\partial x_1}\Big| \; dt\, dx' \;\le\; \frac{1}{2} \int_{\mathbb{R}^d} d\,|\mathrm{sgn}(x_1)\mu|(x)\,. \tag{1}$$

The total flux is given exactly by

$$\int_{[0,\infty[\times\{x_1=0\}} \frac{\partial u}{\partial x_1} \; dt\, dx' \;=\; \frac{1}{2} \int_{\mathbb{R}^d} \mathrm{sgn}(x_1) \; d\mu(x)\,. \tag{2}$$

Formula (2) shows that equality is achieved in (1) when and only when $\mathrm{sgn}\,(x_1)\, d\mu(x)$ is a nonnegative or nonpositive measure which does not charge the hyperplane $\{x_1 = 0\}$.

The key step in the proof is the following special case of (2).

Lemma 1.2. *Denote by $G(t,x) \in C\big([0,\infty[\,;\,\mathcal{S}'(\mathbb{R}^d)\big)$ the fundamental solution of the heat equation, that is the tempered solution with initial value equal to Dirac's delta mass. Then for any $a \neq 0$,*

$$\int_{[0,\infty[\times\{x_1=0\}} G_{x_1}(t,a,x')\ dt\,dx' \;=\; -\frac{1}{2}\,\mathrm{sgn}(a)\,. \tag{3}$$

In addition, the sign of the integrand is equal to the sign of $-a$.

Identity (3) implies both (1) and (2) thanks to the theorems of Fubini and Tonelli.

To see that (3) is physically obvious, note that $-\nabla_x G$ is the heat current from the point source at $x = 0$. For $a < 0$, $G_{x_1}(t,a,x')$ is the heat flux through the hyperplane $x_1 = a$ at the point (t,a,x'). Since the temperature is higher to the right of $\{x_1 = a\}$ than to the left it is reasonable that the flux is always positive. The total flux which is the left hand side of (3) measures the amount of energy that crosses the hyperplane. The fundamental solution has total energy $\int G(t,x)\,dx = 1$. For large times the amount of energy in any slab $|x_1| \le R$ tends to zero. It is physically obvious that for our point source, half the energy moves toward $x_1 = -\infty$ and half toward $x_1 = +\infty$ so the flux through $x_1 = a$ is precisely the first of these two halves. That yields (3).

In [G1], the flux integral in (3) is evaluated directly. This could be done here too. We give an independent evaluation which follows the above intuition.

Proof of Lemma. Define

$$F(a) \;:=\; \int_{[0,\infty[\times\{x_1=a\}} G_{x_1}(t,a,x')\ dt\,dx'\,.$$

The integrand is a smooth function with the same sign as $-a$.

From the scaling identity, $G(\lambda^2 t, \lambda x) = \lambda^{-d} G(t,x)$ it follows that F does not depend on $a < 0$ and we denote its value simply by F. The value of $F(a)$ for $a > 0$ is equal to $-F$ thanks to the symmetry of G with respect to $x_1 = 0$.

When $a < 0$, integrate the heat equation satisfied by G over the half slab $\{x_1 \le a\} \cap \{0 \le t \le T\}$ to find

$$\int_{\{x_1\le a\}\,\cap\,\{0\le t\le T\}} G_t - \Delta G\ dt\,dx \;=\; 0\,.$$

Integrate by parts to obtain

$$\int_{x_1<a} G(T,x)\ dx \;-\; \int_{x_1<a} G(0,x)\ dx \;=\; \int_{[0,T]\times\{x_1=a\}} G_{x_1}(t,a,x')\ dt\,dx'\,.$$

Since $G(0,x) = 0$ in $x_1 \le a$, this yields

$$\int_{x_1<a} G(T,x)\ dx \;=\; \int_{[0,T]\times\{x_1=a\}} G_{x_1}(t,a,x')\ dt\,dx'\,.$$

Denote by $h(T,a)$ the positive quantities on each side of the equality sign. The left hand side is an increasing function of $a < 0$, and, the right hand side is an increasing function of $T > 0$. Thus, h is increasing in both variables, so

$$\lim_{T\to\infty}\ \lim_{a\to 0}\ h(T,a) \;=\; \lim_{a\to 0}\ \lim_{T\to\infty}\ h(T,a)\,. \tag{4}$$

From the right hand side of the definition of h one has $\lim_{T\to\infty} h(T,a) = F(a) = F$. The left hand side of the definition of h implies that

$$\lim_{a\to 0}\ h(T,a) \;=\; \int_{x_1<0} G(T,x)\ dx \;=\; \frac{1}{2}\int_{\mathbb{R}^d} G(T,x)\ dx \;=\; \frac{1}{2}\,,$$

where we have used the symmetry of G. Thus, (4) proves the lemma. ∎

Proof of Theorem. The solution is given by

$$u(t,x) \;=\; \int G(t,x-y)\ d\mu(y)\,,$$

and

$$u_{x_1}(t,x) \;=\; \int G_{x_1}(t,x-y)\ d\mu(y)\,.$$

Therefore

$$\int_{[0,\infty[\times\{x_1=0\}} |u_{x_1}(t,0,x')|\ dt\,dx' \;\le\; \int_{[0,\infty[\times\{x_1=0\}}\int_{\mathbb{R}^d_y} |G_{x_1}(t,(0,x')-y)|\ d|\mu|(y)\,dt\,dx'\,.$$

The right hand integral is evaluated by applying Tonelli's Theorem integrating first with respect to t,x'. From the Lemma, that integral is equal to $1/2$ when $y_1\neq 0$ and is equal to zero when $y_1=0$. Integrating $|d\mu|(y)$ yields (1).
Next evaluate

$$\int_{[0,\infty[\times\{x_1=0\}} u_{x_1}(t,a,x')\ dt\,dx' \;=\; \int_{[0,\infty[\times\{x_1=0\}}\int_{\mathbb{R}^d_y} G_{x_1}(t,(0,x')-y)\ d\mu(y)\,dt\,dx'\,,$$

where the right hand side is absolutely integrable from the preceding computation. Apply Fubini's Theorem integrating first with respect to t,x'. The Lemma implies that the result of that integral is equal to $\mathrm{sgn}(y_1)$ and (2) follows upon integrating $d\mu(y)$. ∎

§2. D'Alembert's wave equation.

§2.1. Preliminaries For real solutions of D'Alembert's wave equation,

$$\Box u = 0\,, \qquad \Box \;:=\; \frac{\partial^2}{\partial t^2} \;-\; \Delta \;:=\; \frac{\partial^2}{\partial t^2} \;-\; \sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}\,, \tag{1}$$

the energy density is $(u_t^2 + |\nabla_x u|^2)/2$ and the differential form of the energy conservation law is

$$0 \;=\; u_t\,\Box u = \partial_t\Big(\frac{u_t^2+|\nabla_x u|^2}{2}\Big) \;-\; \sum_{j=1}^{d}\partial_j(u_t\partial_j u)\,. \tag{2}$$

For complex solutions the result is

$$0 \;=\; \mathrm{Re}\,\Big(\overline{u}_t\,\Box u\Big) = \partial_t\Big(\frac{|u_t|^2+|\nabla_x u|^2}{2}\Big) \;-\; \sum_{j=1}^{d}\partial_j\Big(\mathrm{Re}\,\big(\overline{u}_t\partial_j u\big)\Big)\,. \tag{3}$$

For the energy contained in Ω,

$$E(u,\Omega,t) \ := \ \frac{1}{2}\int_\Omega |u_t(t,x)|^2 + |\nabla_x u(t,x)|^2 \ dx$$

one has the fundamental energy identity

$$E(u,\Omega,b) - E(u,\Omega,a) \ = \ \int_{[a,b]\times\Omega} \nu.\mathrm{Re}\,\overline{u}_t \nabla_x u \ dt\, d\sigma\,.$$

The right hand side is the flux of $\mathrm{Re}\,\overline{u}_t\nabla_x u$ *out* of Ω. The rate of change of the energy in Ω is given by the flux of energy *into* Ω, and, the right hand side is equal to the flux of $-\mathrm{Re}\,\overline{u}_t\nabla_x u$ into Ω, so

$$\text{energy density} \ = \ \frac{|u_t|^2 + |\nabla_x u|^2}{2}\,, \qquad \text{energy current} \ = \ -\mathrm{Re}\,\Big(\overline{u}_t\nabla_x u\Big)\,.$$

The associated velocity for the transport of energy is equal to

$$\frac{\text{energy current}}{\text{energy density}} \ = \ \frac{-2\,\mathrm{Re}\,\big(\overline{u}_t\nabla_x u\big)}{|u_t|^2 + |\nabla_x u|^2}\,.$$

Example. *For the plane wave solutions, $u = e^{i(x\xi + |\xi| t)}$ one has $u_t = i|\xi| u$ and $\partial_j u = i\xi_j u$ so*

$$\text{energy density} = |\xi|^2\,, \qquad \text{and,} \qquad \text{energy current} = -\mathrm{Re}\,\Big(-i|\xi|\overline{u}\ i\xi u\Big) = -|\xi|^2\frac{\xi}{|\xi|}\,. \tag{4}$$

The velocity of energy flow is equal to $-\xi/|\xi|$. This is equal to the group velocity associated with these plane waves. That velocity is usually computed by setting $0 = \nabla_\xi(x\xi + |\xi|t)$. Using the energy flux and energy density gives an alternative computation of the group velocity for conservative systems.

The finite energy solutions are those for which the first partial derivatives of u are continuous function of t with values in $L^2(\mathbb{R}^d)$. For those solutions the total energy

$$E = E(u) \ := \ \int_{\mathbb{R}^d} \frac{|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2}{2} \ dx \tag{5}$$

is independent of t.

Definitions. *The set of finite energy solutions is denoted $\mathcal{H}$. It is a Hilbert space with norm $\|u\|^2_{\mathcal{H}} := E(u)$. Denote by $\mathcal{H}^\infty$ the subset of solutions whose Cauchy data at one (and therefore any) time belong to $C_0^\infty(\mathbb{R}^d)$.*

One of the surprising results in this paper is that for finite energy solutions the absolute energy flux is usually not locally integrable (Thm 2.3.3.**ii**), $\mathrm{Re}\,\overline{u}_t u_{x_1}(t,0,x') \ \notin \ L^1_{\mathrm{loc}}$. This is in sharp contrast with the heat equation flux. Care must be taken in interpreting flux integrals for finite energy solutions. It is well known that the differential energy conservation laws show that the fluxes have traces which, though not integrable, are better behaved than the local regularity of the solutions implies.

Proposition 2.1.1. *If $\Omega \in \mathbb{R}^d$ is an open set lying on one side of its smooth boundary and either ϕ is a uniformly Lipshitzean real valued function of compact support on $\mathbb{R}\times\partial\Omega$, or, $\phi = \chi_{[a,b]}(t)$. Then the bilinear mapping*

$$\mathcal{H}^\infty \times \mathcal{H}^\infty \ \ni\ \{u,v\} \ \mapsto\ \int_{\mathbb{R}\times\partial\Omega} \phi\ \nu.\mathrm{Re}\left(\overline{u}_t\,\nabla_x v + \overline{v}_t\,\nabla_x u\right)\,dt\,d\sigma \ \in\ \mathbb{R}$$

extends uniquely to a continuous map from $\mathcal{H}\times\mathcal{H}$ to $\mathbb{R}$.

Remark 1. For finite energy u the flux given by the above extension is not an integral, but, we will continue to denote it as such.

Remark 2. Taking $u = v$ shows that the flux density $-\nu.\mathrm{Re}\,\overline{u}_t\nabla u$ is a distribution of order 1. That regularity estimate is not sharp. The flux is a distribution of order no higher than 1/2. For example, in the case $\partial\Omega = \{x_1 = 0\}$, finite energy solutions satisfy

$$\mathrm{Re}\left(\overline{u}_t\,\frac{\partial u}{\partial x_1}\right)\ \in\ L^1\big(\mathbb{R}_{x_1}\,;\,L^1_{\mathrm{loc}}(\mathbb{R}_t\times\mathbb{R}^{d-1}_{x'})\big)\,.$$

The energy identity implies that

$$\frac{\partial}{\partial x_1}\,\mathrm{Re}\left(\overline{u}_t\,\frac{\partial u}{\partial x_1}\right)\ \in\ L^1\Big(\mathbb{R}_{x_1}\,;\,\oplus_{\partial\in\{\partial_t,\partial_{x'}\}}\,\partial\,L^1_{\mathrm{loc}}(\mathbb{R}_t\times\mathbb{R}^{d-1}_{x'})\Big)\,.$$

Thus locally the trace belongs an interpolation space

$$\Big[L^1\big(\mathbb{R}_t\times\mathbb{R}^{d-1}_{x'}\big)\,,\ \oplus_{\partial\in\{\partial_t,\partial_{x'}\}}\,\partial\,L^1\big(\mathbb{R}_t\times\mathbb{R}^{d-1}_{x'}\big)\big)\Big]_{1/2}\,,$$

which is 1/2 a derivative smoother than the order 1 estimate.

Proof of Proposition. The proof uses the bilinear differential energy law,

$$0 = \mathrm{Re}\,\Big(\overline{u}_t\,\Box v + \overline{v}_t\Box u\Big) = \partial_t\,\mathrm{Re}\,\Big(\overline{u}_t v_t + \nabla\overline{u}.\nabla v\Big) - \sum_j \partial_j\,\mathrm{Re}\,\Big(\overline{u}_t\partial_j v + \overline{v}_t\partial_j u\Big)\,. \tag{6}$$

To prove (6) consider ∂ equal to either $\partial/\partial t$ or $\partial/\partial x_j$. Then

$$\overline{u}_t\,\partial\partial v \ =\ \partial(\overline{u}_t\,\partial v) - \partial\partial_t\overline{u}\ \partial v\,,\qquad \overline{v}_t\,\partial\partial u\ =\ \partial(\overline{v}_t\,\partial u) - \partial\partial_t\overline{v}\ \partial u\,.$$

Adding yields

$$\overline{u}_t\,\partial\partial v + \overline{v}_t\,\partial\partial u\ =\ \partial\Big(\overline{u}_t\,\partial v + \overline{v}_t\,\partial u\Big) - \Big(\partial\partial_t\overline{u}\ \partial v + \partial\partial_t\overline{v}\ \partial u\Big)\,.$$

Taking the real part, the last term becomes

$$\mathrm{Re}\,\Big(\partial\partial_t\overline{u}\ \partial v + \partial\partial_t\overline{v}\ \partial u\Big)\ =\ \frac{1}{2}\Big(\partial\partial_t\overline{u}\ \partial v + \partial\partial_t u\ \overline{\partial} v + \partial\partial_t\overline{v}\ \partial u + \partial\partial_t v\ \overline{\partial} u\Big)\ =\ \partial_t\,\mathrm{Re}\,(\partial\overline{u}\partial v)\,.$$

Therefore

$$\mathrm{Re}\,\Big(\overline{u}_t\,\partial\partial v + \overline{v}_t\,\partial\partial u\Big)\ =\ \partial\Big(\mathrm{Re}\,(\overline{u}_t\,\partial v + \overline{v}_t\,\partial u)\Big) - \partial_t\Big(\mathrm{Re}\,\partial\overline{u}\ \partial v\Big)\,.$$

Note the special case of $\partial = \partial/\partial t$ where the right hand side is equal to

$$\partial_t \,\mathrm{Re}\Big(\big(\overline{u}_t\, v_t + \overline{v}_t\, u_t\big) \;-\; \overline{u}_t\, v_t \Big) \;=\; \partial_t \,\mathrm{Re}\,\big(\overline{u}_t\, v_t\big)\,.$$

Summing the resulting identities with a minus sign for the $\partial = \partial/\partial x_j$ terms yields (6).

To prove the proposition for Lipshitzean ϕ consider the mapping

$$C_0^\infty(\mathbb{R}\times\partial\Omega)\times\mathcal{H}\times\mathcal{H} \ni \{\phi,u,v\} \mapsto b(\phi,u,v) \;:=\; \int_{\mathbb{R}\times\partial\Omega} \phi\,\nu.\big(\overline{u}_t\,\nabla v + \overline{v}_t\,\nabla u\big)\;dt\,d\sigma\,.$$

It suffices to show that there is a constant $C = C(K)$ so that for all $\phi \in C^\infty(\mathbb{R}\times\partial\Omega)$ with support in K,

$$\big|b(\phi,u,v)\big| \;\le\; C\,E(u)^{1/2}\,E(v)^{1/2}\,\sup\big(|\phi| + |\nabla_{t,x}\phi|\big)\,. \tag{7}$$

If $K \subset \{-T \le t \le T\}$, there is a constant $C_1(K)$ so that for any such ϕ there is $\Phi \in C_0^\infty(\mathbb{R}\times\overline{\Omega})$ with support in $-T\le t\le T$ and satisfying

$$\sup_{\mathbb{R}\times\Omega}\big(|\Phi| + |\nabla_{t,x}\Phi|\big) \;\le\; C_1(K)\,\sup_{\mathbb{R}\times\partial\Omega}\big(|\phi| + |\nabla_{t,x}\phi|\big)\,. \tag{8}$$

Next, multiply the differential energy law (6) by Φ and integrate over $\mathbb{R}\times\Omega$ to obtain

$$0 = \int_{\mathbb{R}\times\Omega} \Phi\Big(e_t + \sum_j \partial_j f_j\Big)\;dt\,dx \;=\; -\int_{\mathbb{R}\times\Omega} \Phi_t\, e + \partial_j\Phi\;f_j\,dt\,dx + \int_{\mathbb{R}\times\partial\Omega} \phi\,\nu.\mathbf{f}\;dt\,d\sigma\,.$$

The last term on the right is exactly $b(\phi,u,v)$. On the other hand using (7) and the Cauchy-Shwartz inequality yield

$$\int_{\mathbb{R}\times\Omega}\Big|\Phi_t\,e + \partial_j\Phi\;f_j\,dt\,dx\Big| \;\le\; C_2(K)\,E(u)^{1/2}\,E(v)^{1/2}\,\sup_{\mathbb{R}\times\Omega}\big(|\Phi| + |\nabla_{t,x}\Phi|\big)\,.$$

This together with (8) proves the desired estimate (7), and the proof of **ii.** is complete.

The case of $\phi = \chi_{[a,b]}(t)$ is easier since integrating (6) over $[a,b]\times\Omega$ followed by an integration by parts yields the energy identity

$$\int_{[a,b]\times\partial\Omega} \nu.\big(\overline{u}_t\,\nabla v + \overline{v}_t\,\nabla u\big)\;dt\,d\sigma \;=\; \int_\Omega \mathrm{Re}\,\big(\overline{u}_t(t)\,v_t(t) + \nabla\overline{u}(t).\nabla v(t)\big)\;d\sigma\Big|_{t=a}^{t=b}\,.$$

The Cauchy-Shwartz inequality implies that the absolute value of the right hand side is no larger than $4\,E(u)^{1/2}\,E(v)^{1/2}$. Thus, $|b(\phi,u,v)| \le 4\,E(u)^{1/2}\,E(v)^{1/2}$ completing the proof. ∎

It is convenient to represent solutions of the wave equation in terms of plane waves, rather than in terms of their Cauchy data. Write

$$u \;=\; (2\pi)^{-d/2}\int a_+(\xi)\;e^{i(x\xi + |\xi|t)}\;d\xi \;+\; (2\pi)^{-d/2}\int a_-(\xi)\;e^{i(x\xi - |\xi|t)}\;d\xi\,.$$

Setting $t = 0$ yields

$$a_+(\xi) + a_-(\xi) \;=\; \hat{u}(0)\,,\qquad i|\xi|\big(a_+(\xi) - a_-(\xi)\big) = \hat{u}_t(0)\,,\qquad a_\pm(\xi) \;:=\; \frac{\hat{u}_t(0) \pm i|\xi|\hat{u}(0)}{2i|\xi|}\,. \tag{9}$$

The group velocity associated with the plane wave $a_\pm(\xi)\,e^{i(x\xi \pm t\tau)}$ is equal to $\mp\xi/|\xi|$. The energy is given by

$$E(u) \;=\; \int |\xi|^2\big(|a_+(\xi)|^2 + |a_-(\xi)|^2\big)\;d\xi\,. \tag{10}$$

The utility of this representation is illustrated by part **ii.** of the following result on the propagation of energy into cones. It is the wave equation analogue of a result of Dollard [D] for Schrödinger's equation.

Proposition 2.1.2. **i.** *If $\Omega \in \mathbb{R}^d$ is a bounded open set and $u \in \mathcal{H}$ is a finite energy solution of the wave equation then as $t \to \pm\infty$ the energy in Ω tends to zero, precisely*

$$\lim_{t \to \pm\infty} \int_\Omega \left(|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2 \right) \, dx \;=\; 0 \,.$$

ii. *Suppose that $\Gamma \in \mathbb{R}^d \setminus 0$ is a conic open set lying on one side of its smooth (except possibly at $\{0\}$) boundary. If $u \in \mathcal{H}$ is a finite energy solution of the wave equation on $\mathbb{R}^{1+d}$ then as $t \to \infty$ (resp. $-\infty$) the energy in Γ converges to the energy in the plane waves whose group velocity belongs to Γ (resp. $-\Gamma$). Precisely,*

$$\lim_{t \to +\infty} \frac{1}{2} \int_\Gamma \left(|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2 \right) \, dx \;=\; \int_{-\Gamma} |\xi|^2 \, |a_+(\xi)|^2 \, d\xi + \int_\Gamma |\xi|^2 \, |a_-(\xi)|^2 \, d\xi \,. \tag{11}$$

Proof. We prove the more interesting result **ii.** first. Since

$$\frac{1}{2} \int_\Gamma \left(|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2 \right) \, dx \;:=\; E(u, \Gamma, t) \;\le\; E(u) \,,$$

it suffices to prove (11) for a dense set of solutions in $\mathcal{H}$.
Consider the dense set with

$$a_\pm \in C_0^\infty \Big(\mathbb{R}^d \setminus \big\{ \overline{\partial\Gamma} \cup \overline{-\partial\Gamma} \big\} \Big) \,.$$

Decompose the resulting solution as the sum $u = u_\Gamma + w$ where

$$u_\Gamma \;:=\; (2\pi)^{-d/2} \int_{-\Gamma} a_+(\xi) \; e^{i(x\xi + |\xi| t)} \; d\xi \;+\; (2\pi)^{-d/2} \int_\Gamma a_-(\xi) \; e^{i(x\xi - |\xi| t)} \; d\xi$$

consists of the part with group velocities in Γ.
The method of nonstationary phase as on page 149 of [R] shows that as $t \to \infty$ and for all α, N

$$\sup_{x \in \Gamma} \big| D^\alpha w(t,x) \big| + \sup_{x \in \mathbb{R}^d \setminus \Gamma} \big| D^\alpha u_\Gamma(t,x) \big| \;\le\; \frac{C(\alpha, N)}{1 + t^N + |x|^N} \,.$$

Therefore

$$\lim_{t \to \infty} E(u, \Gamma, t) \;=\; \lim_{t \to \infty} E(u_\Gamma, \Gamma, t) \;=\; E(u_\Gamma) \;=\; \text{right hand side of (11)} \,,$$

and

$$\lim_{t \to \infty} E(w, \Gamma, t) \;=\; 0 \,.$$

This completes the proof of **ii.**
The proof of **i.** is analogous. The dense set is chosen as those solutions with

$$a_\pm \in C_0^\infty (\mathbb{R}^d \setminus \{0\}) \,.$$

In that case

$$\sup_{x \in \Omega} \big| D^\alpha u(t,x) \big| \;\le\; \frac{C(\alpha, N)}{1 + |t|^N} \,,$$

and **i** follows. ∎

§**2.2. Intuitive results.** In this subsection several results are presented which can be understood by intuitive physical arguments as were the results in §1, and §2.1. In the next subsection, similar reasoning suggests equally reasonable conclusions which are dead wrong.

The first result concerns the flux through the boundary of a smooth compact open set. The intuition is that any ray (or any photon) which enters Ω must leave it (see Figure 1). With a negligible set of exceptions, the number of entries is equal to the number of exits. This suggests that energy which flows in, will flow out, and the total flux should vanish.

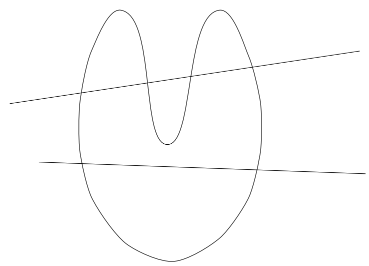

Figure 1. Flux in equals flux out.

Theorem 2.2.1. *If $u \in \mathcal{H}$ is a finite energy solution of the wave equation on $\mathbb{R}^{1+d}$ and Ω is a compact open set lying on one side of its smooth boundary, then the total flux of energy through the boundary of Ω vanishes. Precisely*

$$\lim_{N,M\to\infty} \int_{[-N,M]\times\partial\Omega} \nu.\mathrm{Re}\,\big(\overline{u}_t\,\nabla_x u\big)\; dt\, d\sigma \;=\; 0\,.$$

Proof. The basic energy identity for $u \in \mathcal{H}^\infty$ is

$$E(u,\Omega,M) \;-\; E(u,\Omega,-N) \;=\; \int_{[-N,M]\times\partial\Omega} \nu.\mathrm{Re}\,\big(\overline{u}_t\,\nabla_x u\big)\; dt\, d\sigma\,.$$

Thanks to Proposition 2.1.1, this identity extends by continuity to $u \in \mathcal{H}$.

Finally for $u \in \mathcal{H}$, part **i.** of Proposition 2.1.2 shows that

$$\lim_{t\to\pm\infty} E(u,\Omega,t) = 0\,,$$

proving the Theorem. ∎

The next intuitive result concerns total flux through the boundary of a cone $\Gamma \in \mathbb{R}^d \setminus 0$ (see Figure 2). Rays (or photons) with velocity in Γ enter Γ having crossed the boundary an odd number of times. This gives a net flux of one crossing inward. Those with velocity in $-\Gamma$ lie in Γ for $t << -1$ and leave Γ having crossed the boundary an odd number of times, a net flux of one crossing outward. Except for a negligible set of exceptions, those with other velocities cross the boundary an even number of times suggesting that they don't contribute to the total flux. This motivates the following Theorem.

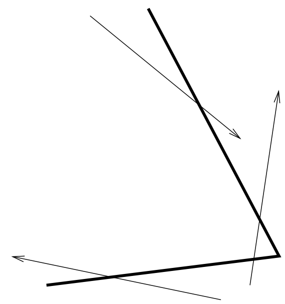

Figure 2. Velocity $\mathbf{v} \in \Gamma$ enters, $\mathbf{v} \in -\Gamma$ leaves, other $\mathbf{v}$'s enter and leave.

Theorem 2.2.2. *Suppose that $\Gamma \in \mathbb{R}^d \setminus 0$ is a conic open set lying on one side of its smooth boundary.*

i. *If $u \in \mathcal{H}$ is a finite energy solution of the wave equation then the total flux through the boundary of Γ is equal to the energy contained in modes with group velocity in Γ minus the energy in modes with velocity in $-\Gamma$, precisely*

$$\lim_{N,M\to\infty} \int_{[-N,M]\times\partial\Gamma} \nu.\mathrm{Re}\left(\overline{u}_t\, \nabla_x u\right)\, dt\, d\sigma$$

is equal to

$$\left(\int_{\xi\in-\Gamma} |\xi|^2\, |a_+(\xi)|^2\, d\xi + \int_{\xi\in\Gamma} |\xi|^2\, |a_-(\xi)|^2\, d\xi\right) - \left(\int_{\xi\in\Gamma} |\xi|^2\, |a_+(\xi)|^2\, d\xi + \int_{\xi\in-\Gamma} |\xi|^2\, |a_-(\xi)|^2\, d\xi\right).$$

ii. *For the flux in $t > 0$,*

$$\lim_{M\to\infty} \int_{[0,M]\times\partial\Gamma} \nu.\mathrm{Re}\left(\overline{u}_t\, \nabla_x u\right)\, dt\, d\sigma \;=\; \int_{\xi_1\in-\Gamma} |a_+(\xi)|^2\, d\xi + \int_{\xi_1\in\Gamma} |a_-(\xi)|^2\, d\xi - E(u)\,.$$

Proof. For **i.**, start with the energy identity

$$\int_{[-N,M]\times\partial\Gamma} \nu.\mathrm{Re}\left(\overline{u}_t\, \nabla_x u\right)\, dt\, d\sigma \;=\; E(u,\Gamma,M) \;-\; E(u,\Gamma,-N)\,.$$

Then part **ii.** of Proposition 2.2.2 completes the proof.

Similarly for **ii.**, one has

$$\int_{[0,M]\times\partial\Gamma} \nu.\mathrm{Re}\left(\overline{u}_t\, \nabla_x u\right)\, dt\, d\sigma \;=\; E(u,\Gamma,M) \;-\; E(u)\,,$$

and again part **ii.** of Proposition 2.1.2 completes the proof. ∎

§**2.3. Counterintuitive results.** This heading is subject to debate because one person's intuition is not necessarily shared by others. We hope to convince you that it is appropriate. Both results to be discussed concern the flux of energy across the hyperplane $\{x_1 = 0\}$.

First false intuition. Consider u a solution of D'Alembert's equation with Cauchy data in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. Suppose that in its plane wave decomposition, only plane waves with group velocities $\mathbf{v}$ with positive first component, $\mathbf{v}_1 > 0$, occur. Since the energy flux for each such plane wave across $\{x_1 = 0\}$ is pointwise positive, the the energy flux of u should likewise be pointwise positive, that is

$$-\mathrm{Re}\left(\overline{u}_t\ \frac{\partial u}{\partial x_1}\right)\ \geq\ 0\,. \tag{12}$$

Remark. The Cauchy data of a solution $u \in \mathcal{H}^\infty \setminus 0$ have compact support so their Fourier Transforms are entire. The supports of $a_\pm$ are equal to all of $\mathbb{R}^d$. Thus, there are no such solutions which are superpositions of just rightward moving plane waves. That is why this intuition is stated for the Schwartz class instead of C_0^∞.

Second false intuition. If you think of flux in terms of adding contributions from rays or photons which cross $\{x_1 = 0\}$ then intergrating $-\mathrm{Re}\left(\overline{u}_t\ \partial u/\partial x_1\right)$ gives positive contributions when the crossing is from left to right and negative for crossings in the other direction. Then $\left|\mathrm{Re}\left(\overline{u}_t\ \partial u/\partial x_1\right)\right|$ counts the crossing always with a positive sign. This counts all crossings and since all but a negligible set of rays cross exactly once this suggests that

$$\lim_{N,M\to\infty}\ \int_{[-N,M]\times\{x_1=0\}} \left|\mathrm{Re}\left(\overline{u}_t\ \frac{\partial u}{\partial x_1}\right)\right|\ dt\,d\sigma\ =\ E(u)\,.$$

This intuition has a fundamental error which is that energy can arrive simultaneously and thereby lead to cancelling fluxes. For example if u is an even function of x_1 then $\partial u/\partial x_1 = 0$ and the left hand side is equal to 0. Taking into account this possibility of cancellation yields the refined intuition

$$\lim_{N,M\to\infty}\ \int_{[-N,M]\times\{x_1=0\}} \left|\mathrm{Re}\left(\overline{u}_t\ \frac{\partial u}{\partial x_1}\right)\right|\ dt\,d\sigma\ \leq\ E(u)\,. \tag{13}$$

Both of these intuitions are correct when the dimension d is equal to one.

Proposition 2.3.1. *Suppose that the dimension $d = 1$ and $u \in \mathcal{H}$ is a finite energy solution of D'Alembert's wave equation.*

i. *Then,* $\mathrm{Re}\,\overline{u}_t(t,0)\ u_x(t,0)$ *is absolutely integrable and*

$$\int_{-\infty}^{\infty} \left|\mathrm{Re}\left(\overline{u}_t\ \frac{\partial u}{\partial x}\right)\right|\ dt\ \leq\ E(u)\,.$$

ii. *If in addition, u is a superposition of plane waves with nonnegative group velocities, precisely* $\mathrm{supp}\, a_\pm \subset \overline{\mathbb{R}}_\mp$, *then*

$$-\mathrm{Re}\left(\overline{u}_t\ \frac{\partial u}{\partial x}\right)\ \geq\ 0\,.$$

Proof. Introduce the characteristic variables

$$u_+ := (\partial_t - \partial_x)u\,, \quad u_- := (\partial_t + \partial_x)u\,, \qquad u_t = \frac{u_+ + u_-}{2}\,, \quad u_x = \frac{u_+ - u_-}{-2}\,.$$

The wave equation is then $(\partial_t \pm \partial_x)u_\pm = 0$. The energy density and energy current are given by

$$\frac{|u_t|^2 + |u_x|^2}{2} \;=\; \frac{|u_+|^2 + |u_-|^2}{4}\,, \qquad \text{and,} \qquad -\mathrm{Re}\left(\overline{u}_t\; u_x\right) \;=\; \frac{|u_+|^2 - |u_-|^2}{4}\,.$$

Therefore,

$$\left| -\mathrm{Re}\left(\overline{u}_t\; u_x\right)\right| \;=\; \left|\frac{|u_+|^2 - |u_-|^2}{4}\right| \;\le\; \frac{|u_+|^2 + |u_-|^2}{4}\,. \tag{14}$$

An immediate consequence of the wave equation in characteristic form is that

$$\int_{-\infty}^{\infty} |u_\pm(t,0)|^2\; dt \;=\; \int_{-\infty}^{\infty} |u_\pm(0,x)|^2\; dt\,. \tag{15}$$

Combining (14) and (15) proves part **i.**

In the one dimensional case, the condition $\mathrm{supp}\, a_\pm \subset \overline{\mathbb{R}}_\mp$ is equivalent to $u_- = 0$. In that case $-\mathrm{Re}\left(\overline{u}_t\; u_x\right) = |u_+|^2/4 \ge 0$ which proves **ii.** ∎

The two intuitions are related. An example demonstrating the falsehood of the first automatically demonstrates the falsehood of the second. To see this, consider a solution which is a superposition of plane waves whose group velocities have positive first component. Then Theorem 2.2.2.**i** with $\Gamma = \{x_1 > 0\}$ implies that

$$\lim_{N,M\to\infty} \int_{[-N,M]\times\{x_1=0\}} - u_t\, \frac{\partial u}{\partial x_1}\; dt\, dx' \;=\; E(u)\,.$$

By hypothesis the integrand is not nonnegative so

$$\int_{[-N,M]\times\{x_1=0\}} \left|u_t\, \frac{\partial u}{\partial x_1}\right| \; dt\, dx' \;-\; \int_{[-N,M]\times\{x_1=0\}} -u_t\, \frac{\partial u}{\partial x_1}\; dt\, dx'$$

is a nondecreasing function of N, M which is strictly positive when N and M are sufficiently large. Therefore for such values,

$$\int_{[-N,M]\times\{x_1=0\}} \left|u_t\, \frac{\partial u}{\partial x_1}\right| \; dt\, dx' \;>\; E(u)\,.$$

This shows that the second intuition is false. We will show that the absolute flux not bounded by $C\,E(u)$ with C as large as one likes. This is much stronger than the fact that the first intuition is false.

At the heart of the errors committed in arriving at the false intuitions is a *sin of superposition.* Both the energy and the energy flux are quadratic quantities and it is dangerous to think of computing them by decomposing a solution as a sum of parts and adding the results. This sin turned out not to be mortal when the dimension $d = 1$. For $d > 1$ the intuitions are dead wrong. We begin with the first.

Theorem 2.3.2. *For each $d > 1$ there is a solution u of D'Alembert's wave equation whose Cauchy data are of Schwartz class, and which is a superposition of plane waves whose group velocities satisfy $\mathbf{v}_1 > 0$, and for which the flux through $\{x_1 = 0\}$ in somewhere negative, i.e. (12) is violated.*

Proof. Start with an example of plane waves.

Example. *Consider*

$$u \;=\; A\,\epsilon\, e^{i(x_2 - t)/\epsilon} \;+\; B\,\epsilon\, e^{i(x_1 - t)/\epsilon} \qquad \text{with} \quad 0 < B < A\,.$$

This is the sum of a plane wave with group velocity $(0, 1, 0 \ldots, 0)$ parallel to the plane $\{x_1 = 0\}$ and a plane wave whose group velocity is equal to $(1, 0, \ldots, 0)$ Then

$$u_t = -i\,A\,e^{i(x_2-t)/\epsilon} - i\,B\,e^{i(x_1-t)/\epsilon}\,, \qquad \frac{\partial u}{\partial x_1} = i\,B\,e^{i(x_1-t)/\epsilon}\,,$$

$$\overline{u}_t\,\frac{\partial u}{\partial x_1} = i\Big(A\,e^{-i(x_2-t)/\epsilon} + B\,e^{-i(x_1-t)/\epsilon}\Big)\,i\,B\,e^{i(x_1-t)/\epsilon}\,.$$

Therefore at at $\{x_1 = 0\}$ the flux is given by

$$-\mathrm{Re}\left(\overline{u}_t\,\frac{\partial u}{\partial x_1}\right) \;=\; B^2 + A\,B\,\cos(x_2/\epsilon)\,.$$

This is not positive. An example with group velocities having strictly positive first component is obtained by perturbation

$$u \;:=\; A\,\epsilon\,e^{i(x_2+\delta x_1-t)/\epsilon} + B\,\epsilon\,e^{i(x_1-t)/\epsilon}\,, \qquad \text{with} \quad 0 < \delta << 1\,.$$

The strategy is to make an elaboration of the example. First construct two families of exact solutions of D'Alembert's equation

$$u_1^\epsilon(t,x) = b(\epsilon,t,x)\; e^{i(x_1 - t/\epsilon)} + O(\epsilon^\infty)\,, \qquad b(\epsilon,t,x) \;\sim\; b_0 + \epsilon\,b_1 + \epsilon^2 b_2 + \cdots\,,$$

$$u_2^\epsilon(t,x) = a(\epsilon,t,x)\; e^{i(x_2 - t/\epsilon)} + O(\epsilon^\infty)\,, \qquad a(\epsilon,t,x) \;\sim\; a_0 + \epsilon\,a_1 + \epsilon^2 a_2 + \cdots\,,$$

More generally

$$v^\epsilon(t,x) \;\sim\; e^{i(x\xi - |\xi|t)/\epsilon}\Big[a_0(x - \xi t/|\xi|) + \epsilon\,a_1 + \epsilon^2 a_2 + \cdots\Big]\,.$$

Here $a_j \in C^\infty(\mathbb{R}^{1+d})$ are determined as follows. Apply the wave operator to the expression $e^{i(x\xi-|\xi|t)/\epsilon}\sum \epsilon^j a_j$ to find $\epsilon^{-1}\, e^{i(x\xi-|\xi|t)/\epsilon}$ times a formal power series in ϵ with smooth coefficients. There is no ϵ^{-2} term because the phase has been chosen as a solution of the eikonal equation. Setting the ϵ^0 coefficient of the power series equal to zero yields the transport equation $\big(\partial_t + \xi/|\xi|.\partial_x\big)a_0 = 0$. Therefore the leading term is of the form $a_0(x - \xi t/|\xi|)$. The initial value of a_0 at $t = 0$ is chosen with support in the unit ball, and identically equal to 1 on a neighborhood of the origin. Thus the leading term a_0 is supported in a tube with foot in the unit ball and generators parallel to $(1, \xi/|\xi|)$.

Setting the coefficients of $\epsilon^1, \epsilon^2, \ldots$ equal to zero yields inhomogeneous transport equations

$$(\partial_t + \frac{\xi}{|\xi|} \cdot \partial_x) a_j = \frac{\Box a_{j-1}}{2\, i\, |\xi|}, \qquad a_j|_{t=0} = 0\,. \tag{16}$$

The support of the a_j are all contained in the tube which supports a_0.

The construction of v^ϵ is as follows. One solves for the a_j. Then using Borel's Theorem one chooses

$$a(\epsilon, t, x) \;\sim\; a_0(x - \xi t/|\xi|) + \epsilon\, a_1 + \epsilon^2 a_2 + \cdots\,,$$

with a supported in the above tube. The residual $r^\epsilon := \Box\big(a\, e^{i(x\xi - |\xi| t)/\epsilon}\big) = O(\epsilon^\infty)$ in C^∞ and is supported in the tube. Then, $v^\epsilon := a\, e^{i(x\xi - |\xi| t)/\epsilon} + c^\epsilon$ where the natural compactly supported $O(\epsilon^\infty)$ corrector c^ϵ is the solution of

$$\Box c^\epsilon = -r^\epsilon, \qquad c^\epsilon|_{t=0} = \partial_t c^\epsilon|_{t=0} = 0\,.$$

The corrector is not supported in the tube.

Let

$$u^\epsilon \;:=\; A\, \epsilon\, u_2^\epsilon + B\, \epsilon\, u_1^\epsilon\,, \qquad \text{with} \quad 0 < B << A\,.$$

The family belongs to $\mathcal{H}^\infty$ and on a neighborhood of the origin $a_0 = b_0 = 1$ so the computation of the example show that the leading term in the asymptotic expansion of the flux is equal to $B^2 + A\, B\, \cos(x_2/\epsilon)$ which is not of one sign.

To complete the proof we modify u^ϵ so that in the plane wave decomposition, the group velocities have $\mathbf{v}_1 > 0$. To accomplish that we analyse the plane wave decomposition of the asymptotic solutions of geometric optics. The point of departure is the factorization

$$\Box \;=\; \big(\partial_t + i|D|\big)\;\big(\partial_t - i|D|\big)\,.$$

This leads to formulas analogous to the one dimensional case,

$$|D|\, u = P_+ u + P_- u\,, \qquad \text{where} \qquad \big(\partial_t \mp i|D|\big) P_\pm u = 0\,,$$

with

$$P_\pm u := (2\pi)^{-d/2} \int e^{i(x\xi \pm |\xi| t)}\, |\xi|\, a_\pm(\xi)\, d\xi = \frac{1}{2}\big(\partial_t \pm i|D|\big) u\,.$$

The parts $P_\pm u$ correspond to the half cones $\tau = \pm|\xi|$ of the characteristic variety. Finite energy solutions are those with $P_\pm u \in C\big(\mathbb{R};\, L^2(\mathbb{R}^d)\big)$, and

$$E(u) \;=\; \|P_+ u(t)\|^2_{L^2(\mathbb{R}^d)} + \|P_- u(t)\|^2_{L^2(\mathbb{R}^d)}\,.$$

Each summand on the right is independent of t.

Lemma 2.3.3. *Suppose that*

$$v^\epsilon \sim e^{i(x\underline{\xi}-|\underline{\xi}|t)/\epsilon}\ \Big[v_0(x-\underline{\xi}t/|\underline{\xi}|)+\epsilon\, v_1+\cdots\Big]$$

is a family of solution of $\Box v^\epsilon = 0$ *with* $v_0 \in C_0^\infty(\mathbb{R}^d)$ *and* $v_j \in C^\infty(\mathbb{R};\, C_0^\infty(\mathbb{R}^d))$ *determined as above. Then for any* $T>0$ *one has uniformly for* $0<\epsilon\le 1$ *and* $|t|\le T$,

i. *The spectrum of* v^ϵ *is strongly localized near* $\xi = \underline{\xi}/\epsilon$ *in the sense that for any* $N>0$

$$|\hat v^\epsilon(t,\xi)| + \epsilon\,|\widehat{P_-v^\epsilon}(t,\xi)| \ \le\ \frac{C(N,T)}{\langle \xi-\underline{\xi}/\epsilon\rangle^N}\,.$$

ii. P_+v^ϵ *is smaller by a factor* ϵ *than* P_-v^ϵ *in the sense that*

$$|\widehat{P_+v^\epsilon}(t,\xi)| \ \le\ \frac{C(N,T)}{\langle \xi-\underline{\xi}/\epsilon\rangle^N}\,.$$

Proof. Observe that for $0<\epsilon\le 1$ and $|t|\le T$,

$$e^{-i(x\underline{\xi}-|\underline{\xi}|t)/\epsilon}\ v^\epsilon \qquad \text{is bounded in} \quad \mathcal{S}(\mathbb{R}^d)\,.$$

Therefore,

$$\langle\xi\rangle^N\ \Big|\mathcal{F}\big(e^{-i(x\underline{\xi}-|\underline{\xi}|t)/\epsilon}\ v^\epsilon\big)\Big| \ \le\ C(N,T)\,.$$

Since $|e^{i|\underline{\xi}|t/\epsilon}|=1$ and $\mathcal{F}\big(e^{-ix\underline{\xi}/\epsilon}\ v^\epsilon\big) \ =\ \mathcal{F}(v^\epsilon)(\xi+\underline{\xi}/\epsilon)$, this implies

$$\langle\xi\rangle^N\ \big|\mathcal{F}(v^\epsilon)(\xi+\underline{\xi}/\epsilon)\big| \ \le\ C(N,T)\,.$$

Making the change of variable $\eta=\xi+\underline{\xi}/\epsilon$ yields the desired estimate for the first summand on the left in **i**.

In exactly the same way one shows that

$$\Big|\mathcal{F}(\epsilon\,\partial_t v^\epsilon)(\xi)\Big| \ \le\ \frac{C(N,T)}{\langle \xi-\underline{\xi}/\epsilon\rangle^N}\,.$$

To estimate $|D|v^\epsilon$ begin with $\mathcal{F}|D|v^\epsilon = |\xi|\,\hat v^\epsilon$ and use the triangle inequality $|\xi|\ \le\ |\underline{\xi}/\epsilon| + |\xi-\underline{\xi}/\epsilon|$ to show that

$$\Big|\mathcal{F}\big(|D|\,v^\epsilon\big)\Big| \ \le\ \frac{1}{\epsilon}\,\frac{C(N,T)}{\langle \xi-\underline{\xi}/\epsilon\rangle^N}\,.$$

Therefore,

$$\Big|\mathcal{F}\big(P_\pm v^\epsilon\big)\Big| \ \le\ \frac{1}{\epsilon}\,\frac{C(N,T)}{\langle \xi-\underline{\xi}/\epsilon\rangle^N}\,,$$

which completes the proof of **i**.

To prove **ii.**, begin by observing that the chain rule shows that

$$e^{-i(x\underline{\xi}-|\underline{\xi}|t)/\epsilon}\Big(\partial_t+\frac{i|\underline{\xi}|}{\epsilon}\Big)v^\epsilon \qquad \text{is bounded in} \quad \mathcal{S}(\mathbb{R}^d)\,.$$

Therefore, as in part **i**,

$$\Big|\mathcal{F}\Big(\partial_t + \frac{i|\underline{\xi}|}{\epsilon}\Big)v^\epsilon\Big| \;\le\; \frac{C(N,T)}{\langle \xi - \underline{\xi}/\epsilon\rangle^N}\,. \tag{17}$$

The inequality

$$\big||\xi| - |\underline{\xi}/\epsilon|\big| \;\le\; \big|\xi - \underline{\xi}/\epsilon\big|$$

implies that

$$\Big|\mathcal{F}\big(|D|v^\epsilon - |\underline{\xi}/\epsilon|v^\epsilon\big)\Big| \;\le\; \langle \xi - \underline{\xi}/\epsilon\rangle\big|\mathcal{F}v^\epsilon\big| \;\le\; \frac{C(N,T)}{\langle \xi - \underline{\xi}/\epsilon\rangle^N}\,,$$

Together with (17) this yields

$$\Big|\mathcal{F}\Big(\partial_t + i|D|\Big)v^\epsilon\Big| \;\le\; \frac{C(N,T)}{\langle \xi - \underline{\xi}/\epsilon\rangle^{N-1}}\,,$$

which is the desired estimate in **ii**. ∎

With this Lemma in hand, we are ready to complete the proof of Theorem 2.3.2 by modifying u^ϵ. The first modification is to turn u_1^ϵ so that it moves at a small angle $0 < \delta << 1$ to the plane $\{x_1 = 0\}$. Introduce the rotation

$$R_\delta(x) \;:=\; (x_1\cos\delta - x_2\sin\delta\,,\, x_1\sin\delta + x_2\cos\delta\,,\, x_3, \dots, x_d)\,,$$

and let

$$w^\epsilon(\delta,t,x) \;:=\; B\,\epsilon\, u_2^\epsilon(t, R_\delta x) + A\,\epsilon\, u_1^\epsilon(t,x)\,.$$

Part **i.** of the Lemma implies that uniformly for $0 < \epsilon < 1$ and $|t| \le T$,

$$\lim_{\delta\to 0}\; \big\|\langle\xi\rangle\mathcal{F}\big(w^\epsilon(\delta,t,x) - u^\epsilon(t,x))\big)\big\|_{L^1(\mathbb{R}^d)} \;=\; 0\,.$$

Therefore

$$\lim_{\delta\to 0}\; \|\nabla_x w^\epsilon - \nabla_x u^\epsilon\|_{L^\infty([-T,T]\times\mathbb{R}^d)} \;=\; 0\,.$$

It follows that

$$\exists\delta_0,\ \forall 0<\delta<\delta_0, \qquad \text{at leading }, O(\epsilon^0), \text{ order, the flux of } w^\epsilon \text{ is not} \ge 0 \text{ near } (0,0)\,.$$

It follows that for these δ, if ϵ is sufficiently small then the flux of $w^\epsilon(\delta,t,x)$ is not positive.

From part **ii.** of the Lemma we know that $\mathcal{F}P_+w^\epsilon = O(\epsilon)$ in $L^1(\mathbb{R}^d_\xi)$ so the flux of P_+w^ϵ is pointwise $O(\epsilon)$. Therefore replacing w^ϵ by $P_-w^\epsilon(\delta,t,x)$ the flux is still nonpositive when ϵ is small.

The spectrum of $u_1^\epsilon(t,x)$ is strongly localized near $(1,0,\dots,0)/\epsilon$ and the spectrum of $u_2^\epsilon(t,x)$ is strongly localized near $R_\delta^{-1}(1,0,\dots,0)/\epsilon = (\sin\delta,\cos\delta,0,\dots)/\epsilon$. Since the first components are strictly positive it follows that the spectrum of $P_-w^\epsilon(\delta,t,x)$ is strongly localized in $\xi_1 \ge c/\epsilon$ Denote by $a_-^\epsilon(\delta,\xi)$ the plane wave coefficient of P_-w^ϵ bearing in mind that its other coefficient vanishes.

Choose a cutoff function $\psi(s)$ smooth in s, supported in $s > 0$ and identically equal to 1 for $s \ge 1$. Define $v^\epsilon(t,x)$ to be the family of solutions of D'Alembert's equation with $P_+v^\epsilon = 0$ and plane wave coefficient of P_-v^ϵ equal to $\psi(\xi_1)\,a_-^\epsilon(\xi)$.

These solutions involve only plane waves with group velocities satisfying $\mathbf{v}_1 > 0$. In addition, the family differs by $O(\epsilon^\infty)$ in $\mathcal{F}L^1$ from the family P_-w^ϵ. Therefore, for ϵ small the functions v^ϵ achieve the desired goal. ∎

The next result shows that (13) is about as false as it possibly could be.

Theorem 2.3.4. i. *For any $T, N > 0$ there is a smooth compactly supported solution $u \in \mathcal{H}^\infty$ of D'Alembert's wave equation so that*

$$\int_{[0,T]\times\{x_1=0\}} \left| \mathrm{Re}\left(\overline{u}_t \, \frac{\partial u}{\partial x_1} \right) \right| \, dt\, dx' \;>\; N\, E(u)\,. \tag{18}$$

ii. *There is a compactly supported finite energy solution $u \in \mathcal{H}$ of D'Alembert's equation so that*

$$\mathrm{Re}\left(\overline{u}_t(t,0,x') \, \frac{\partial u(t,0,x')}{\partial x_1} \right)$$

is not a finite measure on any neighborhood of $(0,0)$. Precisely, for any $\delta > 0$ and $N > 0$ there is a

$$\phi \in C_0^\infty(\{0 < t\} \cap \{x_1 = 0\} \cap \{|t| + |x'| < \delta\}) \qquad \text{with} \qquad \|\phi\|_{L^\infty} \;\le\; 1\,,$$

and

$$\int_{[0,\infty[\times\{x_1=0\}} \phi(t,x')\; \mathrm{Re}\left(\overline{u}_t \, \frac{\partial u}{\partial x_1} \right) \, dt\, dx' \;>\; N\,. \tag{19}$$

Proof. The first remark is that both results follow from the seemingly weaker result that for all $N > 0$

$$\exists u \in \mathcal{H}^\infty, \quad \int_{[0,\infty[\times\{x_1=0\}} \left| \mathrm{Re}\left(\overline{u}_t \, \frac{\partial u}{\partial x_1} \right) \right| \, dt\, dx' \;>\; N\, E(u)\,. \tag{20}$$

To prove that (20) implies **i.** start with a u satisfying (20). Choose $T' > 0$ so that

$$\int_{[0,T']\times\{x_1=0\}} \left| \mathrm{Re}\left(\overline{u}_t \, \frac{\partial u}{\partial x_1} \right) \right| \, dt\, dx' \;>\; N\, E(u)\,. \tag{21}$$

Then define

$$\tilde{u}(t,x) \;:=\; u\Big(\frac{T'}{T}\, t, \, \frac{T'}{T}\, x\Big)\,.$$

The two sides of (21) scale in the same way so, (21) for u is equivalent to (18) for $\tilde{u}$.

Next prove that **i.** implies **ii.** For $n = 1, 2, \ldots$ define

$$x(n) \;:=\; (0,\, 1/2^n\,,\, 0, \ldots, 0) \;\in \mathbb{R}^d\,.$$

For each n use **i.**, translation invariance, and the scaling argument to construct $v_n \in \mathcal{H}^\infty$ with Cauchy data supported in $|x - x(n)| < 1/4^n$ and

$$E(v_n) = 1\,, \qquad \text{and} \qquad \int_{[0,T/8^n]\times\{x_1=0\}} \left| \mathrm{Re}\left(\partial_t \overline{v}(n) \; \frac{\partial v(n)}{\partial x_1} \right) \right| \, dt\, dx' \;>\; 2^n\,.$$

The regions

$$\mathrm{supp}\, v(n) \;\cap\; \Big\{ [0, T/8^n] \times \{x_1 = 0\} \Big\}$$

are disjoint. Choose $N = N(\delta)$ so that $2^{n-1}\delta > 1$, then the unit energy solution

$$u \;:=\; 2^{N-1}\sum_N^\infty \frac{v(n)}{2^n}$$

satisfies **ii**. To show this it suffices to construct $\phi_n \in C_0^\infty(]0,T/8^n[\times\{x_1=0\})$ with sup norm equal to one and

$$\int_{]0,T/8^n[\times\{x_1=0\}} \phi_n(t,x')\ \mathrm{Re}\left(\partial_t\overline{v}(n)\ \frac{\partial v(n)}{\partial x_1}\right)\,dt\,dx' \;>\; \frac{1}{2}\,2^n\,.$$

Then $\phi := \sum_{n=N}^{3N}\phi_n$ satisfies (19).

It remains to prove (20). At the heart of the construction is our example of plane waves.

Example. *When $0 < B << A$ the flux in the example from Theorem 2.3.2 is rapidly oscillating and has absolute value which integrates to a quantity $\sim |AB|$ per unit surface area on $\mathbb{R}_t\times\{x_1=0\}$. Note that the flux AB per unit surface is small compared to $A^2+B^2 \sim A^2$ which is the energy per unit volume so it is not clear that this computation leads to a winning strategy.*

The role of the $\epsilon\, e^{i(x_2-t)/\epsilon}$ family in the example is played by a raylike family

$$u_2^\epsilon(t,x) \;\sim\; \epsilon\, e^{i(x_2-t)/\epsilon}\Big[a_0(x_1,x_2-t,x_3,\dots,x_d) + \epsilon\, a_1 + \epsilon^2 a_2 + \cdots\Big]$$

where for $j\geq 1$ $a_j = a_j(t,x)$ is determined from its initial data $a_j|_{t=0}=0$ by transport equations (17). The leading amplitude is chosen satisfying

$$a_0 \geq 0\,, \qquad \mathrm{supp}\, a_0\big|_{t=0} \;\subset\; \{|x|<1\}\,, \qquad a_0(0) = 1\,. \tag{22}$$

The family u_2^ϵ propagates in a ray tube with speed $\mathbf{e}_2 = (0,1,0,\dots,0)$ parallel to $\{x_1=0\}$. The energy is bounded independent of ϵ.

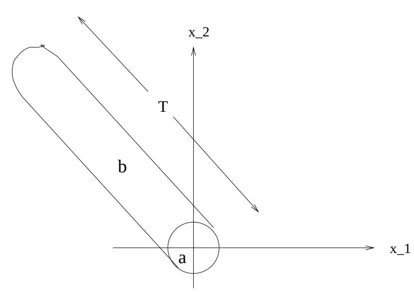

Figure 3. Initial supports of a_0 and b_0

The role of the $\epsilon\, e^{i(x_1-t)/\epsilon}$ family in the example is played by

$$u_1^\epsilon \;\sim\; \epsilon\, e^{i(x_1-t)/\epsilon}\Big[b_0(x_1-t,x_2,x_3,\dots,x_d) + \epsilon\, b_1 + \epsilon^2 b_2 + \cdots\Big]\,,$$

where for $j\geq 1$ the b_j are determined by transport equations of the form (17). The family u_1^ϵ propagates with velocity $\mathbf{e}_1 = (1,0,\dots,0)$ perpendicular to the plane $\{x_1=0\}$.

The strategy is the following. At time $\ell > 0$ the wave u_2^ϵ gives a plane wave supported in a ball about $(\ell, 0, \ell, 0 \ldots, 0)$. We choose the initial data for the family u_1^ϵ so that at $t = 0$ it launches a wave from $(-\ell, \ell, 0, \ldots, 0)$ so as to arrive at $x_1 = 0$ at the time ℓ and interact with the wave u_2^ϵ to give an absolute flux of size $|AB|$ as in the plane wave example.

For fixed $T >> 1$ the initial value of b_0 is chosen with

$$\operatorname{supp} b_0 \ \subset \ \Big\{x \,:\, \operatorname{dist}\big(x, \{x_1 = -x_2,\ 0 \le x_1 \le T\}\big) \le 1\Big\},$$

and

$$b_0(0,x) = 1 \ \text{ when } \ x_1 = x_2, \ \text{and } 0 \le x_1 \le T\,.$$

The initial amplitude b_0 is chosen as a function of $x_1 - x_2$ throughout most of its support so that $E(u_2^\epsilon) \le C\,T$ with C independent of ϵ.

The rays in u_1 move with velocity $\mathbf{e}_1 = (1, 0, \ldots, 0)$ in the x_1 direction. The rays launched at $t = 0$ arrive at the hyperplane $\{x_1 = 0\}$ precisely when the wave u_2 is passing. Let

$$u^\epsilon \ := \ \sqrt{T}\, u_2^\epsilon + u_1^\epsilon\,.$$

By construction,

$$E(u^\epsilon) \le C_1\, T\,, \quad \text{with } C_1 \text{ independent of } \epsilon, T\,. \tag{23}$$

On a neighborhood of width $\delta > 0$ independent of ϵ, T of the line segment

$$\{x_1 = 0 = x_3 = \cdots = x_d\} \ \cap \ \{x_2 = t\} \ \cap \ \{0 \le t \le T\}$$

the leading terms in the waves yield the plane waves in the example with $A \sim \sqrt{T}$ and $B \sim 1$. The flux computation of the example shows that as $\epsilon \to 0$

$$\int_{[0,T]\times\{x_1=0\}} \left|\operatorname{Re}\left(\overline{u}_t\, \frac{\partial u}{\partial x_1}\right)\right| dt\, dx' \ \ge \ C\,T\,A\,B \ \ge \ C_2\, T^{3/2} \quad \text{with } C_2 \text{ independent of } \epsilon, T\,. \tag{24}$$

Taking T large, estimates (23) and (24) yields (20). ∎

Corollary 2.3.5. *Suppose that M is an embedded smooth manifold in $\mathbb{R}^d$ and $\underline{x} \in M$.*

i. *For any $r > 0$ and $N > 0$, there is a smooth compactly supported solution $u \in \mathcal{H}^\infty$ of D'Alembert's wave equation so that*

$$\int_{[0,r]\times B_r(\underline{x})} \Big|\operatorname{Re}\big(\overline{u}_t\, \nu.\nabla_x u\big)\Big| \ dt\, d\sigma \ > \ N\, E(u)\,.$$

ii. *For any $r > 0$ there is a compactly supported finite energy solution $u \in \mathcal{H}$ of D'Alembert's equation so that $\operatorname{Re}\big(\overline{u}_t\, \nu.\nabla_x u\big)$ is not a finite measure on $\{0 < t < r\} \cap M \cap B_r(\underline{x})$.*

Proof. As in the proof Theorem 2.3.4, part **ii** follows from part **i**.

If **i.** were false, then there would exist positive $\underline{r}, \underline{N}$ so that for all $u \in \mathcal{H}^\infty$

$$\int_{[0,\underline{r}]\times B_{\underline{r}}(\underline{x})} \Big|\operatorname{Re}\big(\overline{u}_t\, \nu.\nabla_x u\big)\Big| \ dt\, d\sigma \ \le \ \underline{N}\, E(u)\,. \tag{25}$$

After a translation and rotation we may suppose that $\underline{x} = 0$ and that the tangent plane to M at $\underline{x}$ is the plane $\{x_1 = 0\}$.

For arbitrary $u \in \mathcal{H}$ apply (25) to $u^\epsilon(t,x) := u(\epsilon t, \epsilon x)$ and pass to the limit $\epsilon \to 0$ to conclude that

$$\int_{[0,\infty[\times\{x_1=0\}} \left|\operatorname{Re}\left(\overline{u}_t\, \frac{\partial u}{\partial x_1}\right)\right| \ dt\, dx' \ \le \ \underline{N}\, E(u)\,.$$

Part **i** of Theorem 2.3.4 asserts that no such inequality is valid. This contradiction proves the assertion. ∎

§3. The Shrödinger equation.

The results of §2 are quite general. To illustrate this, we quickly consider the case of Schrödinger's equation

$$u_t \;=\; i\,\Delta\, u\,. \tag{1}$$

Multiplying by $\overline{u}$ and taking twice the real part yields the conservation law (0.6). The conserved density, $|u|^2$, is particle density in quantum mechanical applications, and is wave energy in the diffractive geometric optics approximations to either D'Alembert's equation or Maxwell's equation ([DJMR],[JMR],[BL]),

$$\text{density} \;=\; |u|^2\,, \qquad \text{current} \;=\; 2\,\mathrm{Im}\left(\overline{u}\,\frac{\partial u}{\partial x_j}\right). \tag{2}$$

Example. *The plane wave solutions* $u = e^{i(x\xi - |\xi|^2 t)}$ *have*

$$\text{density} \;=\; 1\,, \qquad \text{current} \;=\; 2\,\xi\,.$$

The corresponding velocity of transport is equal to the quotient 2ξ. *This is equal to the traditional group velocity obtained by setting* $0 = \nabla_\xi (x\xi - |\xi|^2 t)$.

The intuitive results for the wave equation concerning total flux through the boundary of a bounded set vanishing and the flux through the boundary of cones have direct analogues for Shrödinger's equation with analogous proofs. The second assertion is contained in the next Theorem whose proof is omitted.

Theorem 3.1. *Suppose that* $\Gamma \in \mathbb{R}^d \setminus 0$ *is a conic open set lying on one side of its smooth boundary. If* $u \in C\big(\mathbb{R}_t\,;\,L^2(\mathbb{R}^d)\big)$ *is a solution of Schrödinger's equation then the total flux through the boundary of* Γ *is equal to the energy contained in modes with group velocity in* Γ *minus the energy in modes with velocity in* $-\Gamma$*, precisely*

$$\lim_{N,M\to\infty} \int_{[-N,M]\times\partial\Gamma} \nu.2\,\mathrm{Im}\left(\overline{u}\,\nabla_x u\right)\,dt\,d\sigma \;=\; \int_{\xi\in\Gamma} |\hat{u}(t,\xi)|^2\,d\xi - \int_{\xi\in-\Gamma} |\hat{u}(t,\xi)|^2\,d\xi\,.$$

The integrals on the right are independent of t.

First false intuition. For solutions of Shrodinger's equation with $\mathrm{supp}\,\hat{u} \;\subset\; \{\xi_1 > 0\}$ the flux through $\{x_1 = 0\}$ should be nonnegative, that is

$$\mathrm{Im}\left(\overline{u}\,\frac{\partial u}{\partial x_1}\right)\Big|_{\{x_1=0\}} \;\geq\; 0\,.$$

Second false intuition. The absolute value of the flux across $\{x_1 = 0\}$ counts the crossing of particles in either direction so the sum is dominated by the total number of particles, that is

$$\lim_{N,M\to\infty} \int_{[-N,M]\times\{x_1=0\}} \left|2\,\mathrm{Im}\left(\overline{u}\,\frac{\partial u}{\partial x_1}\right)\right|\,dt\,dx' \;\leq\; \int |u(t,x)|^2\,dx\,.$$

The integral on the right is independent of t.

These are just as false as were the analogues for the wave equation in §2.3. The construction of counterexamples to the first rely on plane waves and a WKB construction. The details are omitted. However, the same WKB solutions are used to show the falsehood of the second assertion and we present that construction in detail in the next Theorem.

Theorem 3.2. *For any $T, N > 0$ there is a solution $u \in C^\infty\big(\mathbb{R}_t\,;\,\mathcal{S}(\mathbb{R}^d)\big)$ of Schrödinger's equation so that*

$$\int_{[0,T]\times\{x_1=0\}} \Big|2\,\mathrm{Im}\Big(\overline{u}\,\frac{\partial u}{\partial x_1}\Big)\Big|\;dt\,dx' \;>\; N\int |u(t,x)|^2\;dx\,.$$

The right hand integral is independent of time.

Proof. Under the parabolic scaling $w(t,x) = u(\lambda^2 t, \lambda x)$ which preserves solutions of the Shrödinger equation, both the square of the $L^2(\mathbb{R}^d_x)$ norm and

$$\int_{[0,\infty[\times\{x_1=0\}} \Big|2\,\mathrm{Im}\Big(\overline{u}\,\frac{\partial u}{\partial x_1}\Big)\Big|\;dt\,dx'$$

scale by the same factor λ^{-d}. Thus, to prove the Theorem it suffices to construct a u so that

$$\int_{[0,\infty[\times\{x_1=0\}} \Big|2\,\mathrm{Im}\Big(\overline{u}\,\frac{\partial u}{\partial x_1}\Big)\Big|\;dt\,dx' \;>\; N\int |u(t,x)|^2\;dx\,. \qquad (3)$$

As for the wave equation, the heart of the construction is a sum of plane waves for which the flux is oscillatory. For such a solution, the integral of the absolute value of the flux can be much larger than the integral of the flux.

Example. *With $0 < B << A$ let*

$$u \;:=\; A\,e^{i(x_2-t)} \;+\; B\,e^{i(x_1-t)}\,,$$

the sum of plane waves with $\xi = (0,1,0,\ldots,0)$ and $\xi = (1,0,\ldots,0)$ respectively. Then

$$\Big|2\,\mathrm{Im}\Big(\overline{u}\,\frac{\partial u}{\partial x_1}\Big)\Big|_{\{x_1=0\}} \;=\; 2\Big(B^2 + A\,Be^{-ix_2}\Big)\,.$$

Integrating the flux over a large set one gets $2\,B^2$ per unit surface area. However, when $A >> B$ integrating the absolute value yields AB per unit surface which is much larger. Even this large value is small compared to the density per unit volume $\sim A^2$.

To make use of these plane waves we use asymptotic solutions. The rescaling

$$u(t,x) = v(\epsilon t, \epsilon x)\,, \qquad v(t,x) = u(t/\epsilon\,,\, x/\epsilon),$$

yields

$$\epsilon\, v_t \;-\; \epsilon^2\, i\,\Delta\, v \;=\; 0\,. \qquad (4)$$

The semiclassical form (4) is adapted to the standard WKB *ansatz*

$$v^\epsilon \;\sim\; e^{iS(t,x)/\epsilon}\Big[a_0 + \epsilon a_1 + \ldots\Big]\,, \qquad a_j = a_j(t,x) \in C^\infty\,. \qquad (5)$$

Injecting in the left hand side of (4) yields $e^{iS/\epsilon}$ times a formal power series in ϵ with smooth coefficients. Setting the coefficient of ϵ^0 equal to zero yields the eikonal equation $S_t + |\nabla_x S|^2 = 0$. For our solutions the phase function S will be linear,

$$S = \xi x - |\xi|^2 t\,, \qquad \text{with group velocity} \qquad \mathbf{v} = 2\,\xi\,.$$

Setting the coefficient of ϵ^1 equal to zero yields the transport equation $\big(\partial_t + \mathbf{v}.\partial_x\big)a_0 = 0$. Therefore the leading amplitude has the form $a_0(x - \mathbf{v}t)$ and we take a_0 with compact support so that the leading term is supported in a compact tube of rays $\mathcal{T}$ invariant under the flow of $\partial_t + \mathbf{v}.\partial_x$. The corrector terms a_j with $j \geq 1$ are determined as solutions of

$$\big(\partial_t + \mathbf{v}.\partial_x\big)a_j \;=\; i\,\Delta\, a_{j-1}\,, \qquad a_j\big|_{t=0} = 0\,.$$

Therefore the a_j are supported in the tube supporting a_0. Borel's theorem provides a smooth $a(\epsilon, t, x) \sim \sum \epsilon^j a_j$ supported in $\mathcal{T}$. Injecting $e^{iS/\epsilon}\, a$ into (4) yields a residual which is $O(\epsilon^\infty)$ in C^∞ and the residual is supported in the $\mathcal{T}$. Adding the natural corrector yields a family of exact solutions, v^ϵ of (4) which differs from $e^{iS/\epsilon}\, a$ by $O(\epsilon^\infty)$ in $C^\infty\big(\mathbb{R}_t\,;\,\mathcal{S}(\mathbb{R}^d)\big)$. Rescaling yields a family $u^\epsilon = v^\epsilon(\epsilon t, \epsilon x)$ of exact solutions to Shrödinger's equation which has the form

$$u^\epsilon(t,x) \;\sim\; e^{i(\xi x - |\xi|^2 t)}\Big[a_0\big(\epsilon(x - \mathbf{v}t)\big) + \epsilon a_1(\epsilon t, \epsilon x) + \dots\Big]\,.$$

The approximation is accurate on time intervals $|t| \leq T/\epsilon$ for any fixed $T > 0$.
In this way we construct

$$u_2^\epsilon(t,x) \;\sim\; e^{i(x_2 - t)}\Big[a_0\big(\epsilon(x_1, x_2 + t, x_3, \dots, x_d)\big) + \epsilon a_1(\epsilon t, \epsilon x) + \dots\Big]\,,$$

and

$$u_1^\epsilon(t,x) \;\sim\; e^{i(x_1 - t)}\Big[b_0\big(\epsilon(x_1 + t, x_2, \dots, x_d)\big) + \epsilon b_1(\epsilon t, \epsilon x) + \dots\Big]\,,$$

where the supports at $t = 0$ of a_0 and b_0 are chosen as in Figure 3 with $T >> 1$. Then

$$u^\epsilon \;:=\; \sqrt{T}\, u_2^\epsilon + u_1^\epsilon\,,$$

satisfies as $\epsilon \to 0$

$$\int |u^\epsilon|^2\; dx \;\leq\; \frac{C_1\,T}{\epsilon^d}\,, \qquad \text{with } C_1 \text{ independent of } T, \epsilon\,.$$

The tube supporting u_1 intersects $[0, T/\epsilon] \times \{x_1 = 0\}$ in a set with surface area $\sim T/\epsilon^d$. The calculation of the example shows that the absolute value is $\sim \sqrt{T}$ per unit surface area and one has

$$\int_{[0,T/\epsilon]\times\{x_1=0\}} \left|2\,\mathrm{Im}\left(\overline{u}^\epsilon\,\frac{\partial u^\epsilon}{\partial x_1}\right)\right|\; dt\, dx' \;\geq\; \frac{C_2\,T^{3/2}}{\epsilon^d}\,, \qquad \text{with } C_2 \text{ independent of } T, \epsilon\,.$$

Given $N > 0$ choose T so large that

$$\frac{C_2\,T^{3/2}}{C_1 T} \;>\; N\,.$$

Then for ϵ sufficiently small, the solution u^ϵ satisfies (3), and the proof is complete. ∎

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