Hyperbolic L^p Multipliers are Translations

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Abstract. We prove that the only constant coefficient hyperbolic systems $L = \partial_t + \sum_{j=1}^d A_j\,\partial_j$ whose solution operators are continuous on L^p for some $p \neq 2$ are those for which the matrices A_j are simultaneously diagonalisable.

§1. Introduction.

Consider

$$L = \partial_t + \sum_{j=1}^d A_j\,\partial_j\,, \qquad \partial_j \;:=\; \frac{\partial}{\partial x_j}\,, \qquad A_j \in \mathrm{Hom}(\mathbb{C}^N)\,. \tag{1}$$

This is the general constant coefficient homogeneous first order system of partial differential operators with noncharacteristic initial surface $\{t = 0\}$. The initial value problem is

$$Lu \;=\; 0\,, \qquad u(0,x) \;=\; f\,.$$

Introduce the symbol

$$A(\xi) \;:=\; \sum_{j=1}^d A_j\,\xi_j\,.$$

The associated evolution operators which map $u(0,\cdot)$ to $u(t,\cdot)$ is the family of Fourier multipliers $e^{-itA(\xi)}$,

$$e^{-tA(\partial)} \;:=\; \mathcal{F}^*\; e^{-itA(\xi)}\;\mathcal{F}\,. \tag{2}$$

Our main result shows that boundedness in L^p for $p \neq 2$ is very rare. The evolution operator is L^p bounded if and only if it consists of simple translations.

If the multiplier (2) is an L^p multiplier then it is an L^q multiplier for the dual index, $p^{-1}+q^{-1} = 1$. By interpolation it is an L^2 multiplier so

$$\exists C, \quad \forall \xi \in \mathbb{R}^d, \qquad \big\| e^{-iA(\xi)} \big\|_{\mathrm{Hom}(\mathbb{C}^N)} \;\le\; C\,.$$

In particular, for fixed real ξ the family $e^{-itA(\xi)} = e^{-iA(t\xi)}$ is uniformly bounded in t. Therefore $A(\xi)$ has only real eigenvalues and no nontrivial Jordan blocks. Equivalently, $A(\xi)$ is similar to a real diagonal matrix. The system L is called **hyperbolic** if and only if the matrices $A(\xi)$ have real spectrum. Therefore, hyperbolicity is a necessary to generate L^p multipliers.

The characteristic variety of L is defined to be

$$\mathrm{Char}\, L \;:=\; \Big\{(\tau,\xi) \in \mathbb{R}^{1+d} \setminus 0 \;:\; \det(\tau I + A(\xi)) = 0\Big\}\,.$$

It is a conic real algebraic variety. For real ξ, the roots τ are the negatives of the eigenvalues of $A(\xi)$, so are real. Thus, over each point ξ there is at least one point and at most N points in the variety. Thus the variety is of dimension d.

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With the exception of an at most $d-1$ dimensional conic singular set, the characteristic variety is locally an embedded real analytic hypersurface (see for example [BR]). The conormal directions to smooth points of the variety are lines in (t,x) whose speeds are, by definition, the **group velocities** (see [AR, §1.2] or the web version of [R]). For a curved sheet of the characteristic variety, the conormal directions sweep out a variety of dimension greater or equal to 1, so there is a continuum of group velocities and the associated solutions tend to spread out and decay. The next definition singles out the extreme opposite case where the variety has no curved sheets and there is no dispersion of this sort.

Definition. *A hyperbolic system L is called* **nondispersive** *if and only if its characteristic variety is a finite union of hyperplanes.*

For a nondispersive system, denote by κ the number of distinct hyperplanes in the characteristic variety of L. Each hyperplane has an associated group velocity $\mathbf{v}_\mu$ and the equation of the plane is $\tau + \mathbf{v}_\mu.\xi = 0$. The characteristic variety is then

$$\cup_{\mu=1}^{\kappa} \Big\{ (\tau, \xi) \ : \ \tau + \mathbf{v}_\mu.\xi = 0 \Big\}.$$

The eigenvalues of $A(\xi)$ are the linear functions $\lambda_\mu(\xi) = -\mathbf{v}_\mu.\xi$ with $1 \le \mu \le \kappa$. The nondispersive systems are exactly those for which the eigenvalues of $A(\xi)$ are linear functions of $\xi \in \mathbb{R}^d$.

Theorem. The following are equivalent.

i. The multiplier (2) is for some $t \neq 0$ and $2 \neq p \in [1, \infty]$ an L^p multiplier.

ii. The matrices $e^{-iA(\xi)}$ are uniformly bounded for $\xi \in \mathbb{R}^d$ and the system is nondispersive.

iii. The A_j are commuting diagonalizable matrices with real spectrum.

iv. The multiplier (2) is for all $t \neq 0$ and all $1 \le p \le \infty$ an L^p multiplier.

Remarks. 1. The scalar case, $N = 0$, is trivial. The one dimensional case, $d = 1$, is elementary.

2. The case of hermitian symmetric A_j is proved by P. Brenner [B] who introduced a two step strategy of proof, **i** $\Rightarrow$ **ii** $\Rightarrow$ **iii** which we follow. In the symmetric case, the implication **ii** $\Rightarrow$ **iii** is Theorem 2 of Motzkin and Tausky [MT]. Our implication **ii** $\Rightarrow$ **iii** is a generalization of their result to the nonsymmetric case.

3. If the evolution is an L^p multiplier for some p then the operator remains hyperbolic for all lower order perturbations by Duhamel's construction. In the case $N = 2$, a theorem of Strang [S, Appendix], shows that for such strongly hyperbolic systems, a change of variable in $\mathbb{C}^2$ simultaneously symmetrizes the matrices A_j. Brenner's result then applies. Thus the case $N = 2$ is a consequence of known results.

4. If the A_j are all upper triangular with real diagonal, then the system L is hyperbolic and nondispersive. The Theorem shows that if the A_j do not commute, the time evolution is not an L^2 multiplier even though the dispersion relations are trivial.

5. If **iii** holds we can by changing coordinates in $\mathbb{C}^N$ suppose that the A_j are all real diagonal matrices. In that case, for each $1 \le i \le d$ let

$$\mathbf{w}_i \ := \ (A_{1,ii}, \dots, A_{d,ii})$$

be the d-vector constructed from the i^{th} diagonal elements of the matrices A_j. Then, the solution of the Cauchy problem $Lu = 0$ with initial value

$$u(0,x) \ = \ f(x) \ = \ \Big(f_1(x), \dots, f_N(x)\Big)$$

is given by

$$u(t,x) \;=\; \Big(f_1(x-\mathbf{w}_1 t),\dots,f_N(x-\mathbf{w}_N t)\Big).$$

These independent translations are isometries in L^p for all p so that **iv.** holds. Each of the speeds $\mathbf{w}_j$ is equal to one of the group velocities $\mathbf{v}_\mu$.

6. That **iv** implies **i** is trivial.

The difficult implications are **i** $\Rightarrow$ **ii** and **ii** $\Rightarrow$ **iii**. They are proved in §2. and §3 respectively. Our §3 bears a family resemblance to the results and proofs in §2.5 of [K]. They seem truly independent.

§2. Proof of i $\Rightarrow$ ii.

The key to showing that the propagator is not an L^p multiplier for $p \neq 2$, is to show that there are solutions which at $t=0$ are spread over a ball of fixed radius, while for large times t they are spread over a set whose measure $m(t)$ tends to infinity. If the initial amplitudes are of order 1, then respect for the norm L^2 implies that the typical amplitude $a(t)$ at time t satisfies

$$\|u(t)\|^2_{L^2} \;\sim\; |a(t)|^2\; m(t) \;\sim\; 1,$$

So $a(t) \sim m(t)^{-1/2}$ and

$$\|u(t)\|^p_{L^p} \;\sim\; |a(t)|^p\; m(t) \;\sim\; m(t)^{1-p/2}\,.$$

For $1 \le p < 2$, the L^p norm tends to infinity. Since the multiplier norm is independent of t so this is sufficient to conclude that $e^{-itA(\xi)}$ is not an L^p multiplier.

The construction we present is elementary and self contained. It uses geometric optics with nonlinear phases. The spread of rays accounts for the increase in $m(t)$. The shortest proof that we know, suggested by J.-M. Bony, is less elementary. It uses conormal solutions associated to a lagrangian which has simple projection over $t=0$ and has a fold in the future. One applies sharp theorems identifying the (different) memberships in L^p in the initial hyperplane and under the fold. The details of the construction of a lagrangian with a fold, and a solution with nonvanishing symbol at fold points, are left to the interested reader. A third argument uses diffractive geometric optics with linear phases. The discontinuity in L^p then follows from the fact that the linear Schrödinger evolution is discontinuous in L^p.

Proof that i $\Rightarrow$ ii. If the family of matrices $e^{-iA(\xi)}$ is not uniformly bounded then it is not an L^2 multiplier and therefore not an L^p multiplier for any p. Thus we need to show that if the family is uniformly bounded and the system is not nondispersive then **i.** is violated.

The smooth sheets of the characteristic variety are given locally by equations $\tau + \lambda(\xi) = 0$ with real analytic λ. Choose a point $\underline{\xi} \neq 0$ on a smooth sheet with equation defined by λ and so that $\lambda_{\xi\xi}(\underline{\xi})$ has rank $m > 0$.

By a linear change of coordinates in $\mathbb{R}^d$ we may suppose that $\underline{\xi} = (1,0,\dots,0)$. By an orthogonal change of coordinates in $\mathbb{C}^N$ we may suppose that

$$\lambda_{\xi\xi}(\underline{\xi}) \;=\; \operatorname{diag}\Big(\nu_1,\nu_2,\dots,\nu_m,0,\dots,0\Big)\,, \qquad \nu_j \neq 0\,. \tag{2.1}$$

Define

$$\phi_0(x) \;:=\; x_1 \;+\; \sum_{j=1}^m \frac{x_j^2}{2\,\nu_j}\,, \qquad \nabla_{xx}\phi_0 = \operatorname{diag}\Big(\frac{1}{\nu_1},\dots,\frac{1}{\nu_m},0,\dots,0\Big)\,. \tag{2.2}$$

The phase $\phi(t,x)$ is a solution of the eikonal equation

$$\phi_t + \lambda(\nabla_x \phi) \;=\; 0\,, \qquad \phi(0,x) = \phi_0(x)\,. \tag{2.3}$$

Hamilton-Jacobi theory shows that both ϕ and $\nabla_x \phi$ are constant on the **rays** which start at points $(0,x)$ and have velocity given by $\big(1, \lambda_\xi(\nabla_x \phi_0(x))\big)$ that is the points

$$\Big(t\,,\; x + t\,\lambda_\xi\big(\nabla_x \phi_0(x)\big)\Big) \;:=\; \Big(t, \Phi(t,x)\Big)\,. \tag{2.4}$$

Equation (2.4) implies,

$$D_x \Phi(t,x) \;=\; I + t\lambda_{\xi\xi}\big(\nabla_x \phi(x)\big)\; \nabla_{xx}\phi_0(x)\,.$$

Equations (2.1) and (2.2) imply

$$\lambda_{\xi\xi}\big(\nabla_x \phi(x)\big)\; \nabla_{xx}\phi_0(x) \;=\; \mathrm{diag}\,\big(1,1,\dots,1,0,\dots,0\big) + O(|x|)\,,$$

with m diagonal ones. Denote $w = (w^I, w^{II})$ the first m components and the rest. Since $\nabla_{xx}\phi$ kills the last components we have for $|x| \le \rho$,

$$\Big\langle D_x \Phi(t,x) w, w \Big\rangle \;\ge\; \big(1 + t - O(t\rho)\big)|w^I|^2 + \big(1 - O(t\rho)\big)|w^{II}|^2 - O(t\rho)|w^I||w^{II}|\,.$$

Choose positive γ sufficiently small so that if

$$T_\rho \;:=\; \frac{\gamma}{\rho}\,,$$

then for all $t \in [0, T_\rho]$ each of the three $O(t\rho)$ is no larger than $1/4$. Then,

$$\forall\; 0 \le t \le T_\rho\,, \quad \forall\; |x| \le \rho\,, \qquad \Big\langle D_x \Phi(t,x) w, w \Big\rangle \;\ge\; \frac{(1+t)|w^I|^2 + |w|^2}{2}\,. \tag{2.5}$$

Lemma. *If a real matrix M and a strictly positive definite real symmetric matrix R satisfy $\langle Mw, w\rangle \ge \langle Rw, w\rangle$ for all $w \in \mathbb{R}^N$, then $\det M \ge \det R$.*

Proof of lemma. The substitution $\tilde{w} = R^{1/2} w$ reduces to the case $R = 1$. The Cauchy-Schwarz inequality then implies that $\|Mw\| \ge \|w\|$ for real vectors and therefore that $\|M^{-1}w\| \le \|w\|$ for real vectors. For complex vectors express in terms of their real and imaginary parts to find

$$\|M^{-1}(u + iv)\|^2 \;=\; \|M^{-1}(u)\|^2 + \|M^{-1}(v)\|^2 \;\le\; \|u\|^2 + \|v\|^2 \;=\; \|u + iv\|^2\,.$$

Thus M^{-1} has norm no larger than 1, so the eigenvalues of M are all of magnitude at least 1. Therefore $|\det M| \ge 1$.

The segment $(1-\theta)M + \theta I$ for $0 \le \theta \le 1$ connects M to the identity by invertible real matrices so $\det M > 0$. Therefore $\det M \ge 1$ which completes the proof in the case $R = I$. ∎

Apply the lemma (2.5) to conclude

$$\forall\; 0 \le t \le T_\rho, \quad \forall\; |x| \le \rho, \qquad \det D_x \Phi(t,x) \;\ge\; \frac{(1+t)^m}{2^N} \tag{2.6}$$

Taylor's formula,

$$\Phi(t,x) - \Phi(t,y) \ = \ \int_0^1 D_x\Phi(t, y+\theta(x-y)).(x-y)\ d\theta\,,$$

implies that

$$\forall\ 0 \le t \le T_\rho,\ |x| \le \rho,\ |y| \le \rho\,, \quad \Big\langle \Phi(t,x) - \Phi(t,y)\,,\, x-y \Big\rangle \ \ge\ \frac{1}{2}\Big((1+t)|x^I - y^I|^2 + |x^{II} - y^{II}|^2\Big)\,.$$

In particular, for all $0 \le t \le T_\rho$ the map $x \to \Phi(t,x)$ is a bijective diffeomorphism from $\{|x| < \rho\}$ onto its image.

Denote by Ω_ρ the relatively open subset of $\{0 \le t \le T_\rho\} \times \mathbb{R}^d$ swept out by the images of $\{|x| < \rho\}$ by these diffeomorphisms. It is the tube of forward rays whose feet at $t = 0$ belong to the ball of radius ρ. The recipe of Hamilton and Jacobi provides a phase function ϕ which is defined and smooth on Ω_ρ. On the intersections $\Omega_\rho \cap \Omega_{\rho'}$ the phases agree. As ρ decreases, the tube becomes narrower about the ray through the origin, and penetrate further into the future. Typically the phase ϕ will develop caustics outside these narrow tubes well before the time T_ρ. We work in the tubes where ϕ defines a long lived phase.

On Ω_ρ define the group velocity field

$$\mathbf{v}(t,x) = \lambda_\xi(\nabla_x\phi(t,x))\,. \tag{2.7}$$

The rays $t \to (t, \Phi(t,x))$ from (2.4) are integral curves of the vector field $\partial_t + \mathbf{v}.\partial_x$. Equivalently, Φ is the flow generated by $\mathbf{v}$,

$$\frac{d}{dt}\,\Phi(t,x) \ = \ \mathbf{v}(\Phi(t,x))\,, \qquad \Phi(0,x) = x\,.$$

Differentiating with respect to x yields the evolution of the deformation $D_x\Phi(t,x)$ and the Jacobian determinant $J(t,x) := \det\big(D_x\Phi(t,x)\big)$,

$$\frac{d}{dt}\,D_x\Phi(t,x) \ = \ \big(D_x\mathbf{v}\big)(\Phi(t,x))\ D_x\Phi(t,x)\,, \qquad \frac{d}{dt}\,J(t,x) \ = \ (\operatorname{div}\mathbf{v})(\Phi(t,x))\ J(t,x)\,. \tag{2.8}$$

We construct solutions of $Lu = 0$ in the form $e^{i\phi/\epsilon}[a_0 + \epsilon a_1 + \ldots]$ with $a_j(t,x)$ smooth and supported in Ω_ρ. Toward this end compute

$$L\left(e^{i\phi/\epsilon}[a_0 + \epsilon a_1 + \ldots]\right) \ = \ \frac{1}{\epsilon}\,e^{i\phi/\epsilon}\Big[b_0 + \epsilon b_1 + \epsilon^2 b_2 + \ldots\Big]\,,$$

with

$$\begin{aligned} b_0 \ &= \ iL(\phi_t, \nabla_x\phi)\,a_0\,, \\ b_1 \ &= \ iL(\phi_t, \nabla_x\phi)\,a_1 + L(\partial)a_0\,, \\ b_j \ &= \ iL(\phi_t, \nabla_x\phi)\,a_{j+1} + L(\partial)a_j\,, \quad j \ge 1\,. \end{aligned} \tag{2.9}$$

Since ϕ is a solution of the eikonal equation (2.3), the matrix $L(\phi_t(t,x), \nabla_x\phi(t,x))$ is singular for all $(t,x) \in \Omega$.

Since $A(\xi)$ are diagonalisable for all ξ, it follows that $L(\phi_t(t,x), \nabla_x\phi(t,x)) = \phi_t I + A(\nabla_x\phi)$ is diagonalisable for all $(t,x) \in \Omega$. Denote by $\pi(t,x)$ the spectral projection onto the kernel of $L(\phi_t(t,x), \nabla_x\phi(t,x))$ and by $Q(t,x)$ the partial inverse defined by

$$Q(t,x)\,\pi(t,x) \;=\; 0\,, \qquad Q(t,x)\,L(\phi_t(t,x), \nabla_x\phi(t,x)) \;=\; I - \pi(t,x)\,. \tag{2.10}$$

Since we are working on a smooth sheet of the characteristic variety, the contour integral representations of $\pi(t,x)$ and $Q(t,x)$ show that they are smooth functions of $(t,x) \in \Omega$.

The equation $b_j = 0$ is equivalent to the pair of equations $\pi(t,x)b_j = 0$ and $Q(t,x)b_j = 0$. The equation $\pi b_0 = 0$ is automatic, so the equations $b_0 = 0$ and $b_1 = 0$ are equivalent to the trio of equations

$$\pi(t,x)a_0 = a_0\,, \qquad \pi(t,x)\,L(\partial)\,a_0 = 0\,, \qquad (I - \pi(t,x))a_1 = iQ(t,x)L(\partial)a_0\,. \tag{2.11}$$

Lemma. *For functions $w(t,x)$ which satisfy the polarization $\pi(t,x)w = w$, one has*

$$\pi(t,x)\,L(\partial)\,w \;=\; \Big(\partial_t + \mathbf{v}(t,x).\partial_x + \frac{1}{2}\,\mathrm{div}\,\mathbf{v}(t,x)\Big)w\,. \tag{2.12}$$

The calculation uses second order perturbation theory in the following form (see [K, formulas (II.2.13), (II.2.33)]). This part of perturbation theory corresponds to the fundamental algebraic lemmas of geometric optics.

Definition. *An eigenvalue λ of a matrix A is semisimple when the kernel and range of $A - \lambda I$ are complementary subspaces. In this case denote by π the spectral projection onto the kernel of $A - \lambda I$ along its range and by Q the partial inverse defined by*

$$Q\,\pi \;=\; 0\,, \qquad Q\,\big(A - \lambda\,I\big) = I - \pi\,.$$

Proposition. *Suppose that $]a,b[\ni s \to A(s)$ is a smooth family of complex matrices with an isolated smooth semisimple eigenvalue $\lambda(s)$. Then $\lambda(s)$ and $\pi(s)$ are smooth functions of s whose first derivatives satisfy*

$$\lambda'(s)\,\pi(s) \;=\; \pi(s)\,A'(s)\,\pi(s)\,, \quad \lambda''\,\pi \;=\; \pi\,A''\,\pi \,-\, 2\,\pi\,A'\,Q\,A'\,\pi\,, \quad \pi' \;=\; -\pi\,A'\,Q - QA'\pi\,. \tag{2.13}$$

Proof of Lemma. In $\pi\,L(\partial)\,w$ write the spatial derivatives as

$$\pi\,A_j\partial_j w \;=\; \pi\,A_j\partial_j\,(\pi w) \;=\; \pi\,A_j\,\pi\,\partial_j w + \pi\,A_j\,(\partial_j\pi)\,\pi\,w\,. \tag{2.14}$$

Consider the eigenvalue $\lambda(\xi)$ and eigenprojection $\pi(\xi)$ of the matrix $A(\xi)$ as functions of the parameter ξ_j. The first formula from perturbation theory and (2.7) imply that

$$\pi\,A_j\,\pi \;=\; \partial\lambda/\partial\xi_j\,\pi \;=\; \mathbf{v}_j\pi\,. \tag{2.15}$$

Since the second derivatives of $A(\xi)$ vanish, the formula for λ'' implies, after depolarization, that

$$\frac{\partial^2 \lambda}{\partial \xi_j \partial \xi_k}\, \pi \;=\; -\pi A_j Q A_k \pi \;-\; \pi A_k Q A_j \pi. \tag{2.16}$$

Next consider the eigenvalue $\lambda(\nabla_x \phi)$ and eigenprojections $\pi(t,x)$ of $M(t,x) := \sum_k A_k \partial_k \phi(t,x)$ as functions of the parameter x_j. The perturbation formula yields

$$(\partial_j \pi)\, \pi \;=\; -Q\, \partial_j M\, \pi \;=\; -Q \sum_k A_k\, \pi\, \frac{\partial^2 \phi}{\partial x_k \partial x_j}\,.$$

Therefore using (2.16) yields,

$$\pi\, A_j\, (\partial_j \pi)\pi = -\sum_k \pi A_j Q A_k \pi\, \frac{\partial^2 \phi}{\partial x_k \partial x_j}$$

and

$$\sum_j \pi\, A_j\, (\partial_j \pi)\pi = -\sum_{j,k} \pi A_j Q A_k \pi\, \frac{\partial^2 \phi}{\partial x_k \partial x_j} = \frac{1}{2}\big(\operatorname{div} \mathbf{v}\big)\, \pi \tag{2.17}$$

Combining (2.14, 2.15, 2.17) yields when $w = \pi\, w$,

$$\pi\, L(\partial) w \;=\; \pi\, \big(\partial_t + \mathbf{v}.\partial_x\big) w + \frac{1}{2}\, \big(\operatorname{div} \mathbf{v}\big) w\,.$$

Since $\nabla_x \phi$ is constant along integral curves of $\partial_t + \mathbf{v}.\partial_x$ it follows that $\pi(t,x)$ is also constant so

$$\pi\, \big(\partial_t + \mathbf{v}.\partial_x\big) w \;=\; \big(\partial_t + \mathbf{v}.\partial_x\big)\, \pi w \;=\; \big(\partial_t + \mathbf{v}.\partial_x\big) w\,,$$

and the proof of the lemma is complete. ∎

The equation for $a_0 = \pi\, a_0$ is therefore the transport equation,

$$\big(\partial_t + \mathbf{v}.\partial_x + \frac{1}{2}\operatorname{div} \mathbf{v}\big) a_0 = 0\,, \tag{2.18}$$

along the rays sweeping out Ω_ρ. Therefore, a_0 is determined once one prescribes

$$a_0(0,x) \in C_0^\infty(\{|x| < \rho\}) \setminus \{0\}\,, \qquad \text{with} \quad \pi(0,x) a_0(0,x) = a_0(0,x)\,. \tag{2.19}$$

Setting $b_j = 0$ for $j \geq 1$ yields the equations

$$\big(\partial_t + \mathbf{v}.\partial_x + \frac{1}{2}\operatorname{div} \mathbf{v}\big)\big(\pi\, a_j\big) x = -\pi\, L(\partial)\, \big((I - \pi) a_j\big)\,, \qquad (I - \pi) a_{j+1} = iQ\, L(\partial)\, a_j\,. \tag{2.20}$$

Impose the initial condition

$$\pi\, a_j(0,x) \;=\; 0\,, \qquad j \geq 1\,. \tag{2.21}$$

This suffices for a recursive determination of smooth profiles a_j which are supported in Ω_ρ.

Borel's Theorem constructs a smooth function $a(\epsilon,t,x)$ supported in $[0,1[\times\Omega_\rho$ so that for small ϵ,

$$a(\epsilon,t,x)\ \sim\ \sum_j \epsilon^j\,a_j\,,\qquad \text{in}\quad C^\infty_{(0)}([0,T_\rho]\times\mathbb{R}^d)\,.$$

Then

$$L(\partial)\Big(e^{i\phi(t,x)/\epsilon}\,a(\epsilon,t,x)\Big)\ :=\ r^\epsilon(t,x)\ =\ O(\epsilon^\infty)\qquad \text{in}\quad C^\infty_{(0)}([0,T_\rho]\times\mathbb{R}^d)\,.$$

Define a corrector c^ϵ by

$$L(\partial)\,c^\epsilon\ =\ -r^\epsilon\,,\qquad c^\epsilon(0,x)=0\,.$$

Since $L(\partial)$ is L^2 well posed it follows that for all $T>0$

$$c^\epsilon\ =\ O(\epsilon^\infty)\qquad \text{in}\quad C^\infty_{(0)}([0,T_\rho]\times\mathbb{R}^d)\,.$$

Define

$$u^\epsilon\ :=\ e^{i\phi(t,x)/\epsilon}\,a(\epsilon,t,x)\ +\ c^\epsilon(t,x)\,. \tag{2.22}$$

Then $Lu^\epsilon=0$.

We will compare the L^p norm of u^ϵ at time $t=T_\rho$ to that at time $t=0$ when ϵ is small. From the form of u^ϵ one has for any $t\in[0,T_\rho]$,

$$\lim_{\epsilon\to 0}\ \|u^\epsilon(t)\|^p_{L^p(\mathbb{R}^d)}\ =\ \int |a_0(t,y)|^p\ dy\,. \tag{2.23}$$

Equations (2.8) and (2.18) yield an explicit computation of a_0.

Lemma. *The amplitude a_0 satisfies the transport equation (2.18) if and only if the function*

$$a_0(t,\Phi(t,x))\,\sqrt{J(t,x)}$$

does not depend on t.

Proof of lemma. Equation (2.8) implies that $\sqrt{J(t,x)}$ satisfies

$$\partial_t\sqrt{J(t,x)}\ =\ \frac{1}{2\,\sqrt{J(t,x)}}\ \partial_t J\ =\ \frac{1}{2}\ \big(\mathrm{div}\,\mathbf{v})(\Phi(t,x)\big)\ \sqrt{J(t,x)}\,.$$

Therefore

$$\partial_t\Big(a_0(t,\Phi(t,x))\,\sqrt{J(t,x)}\Big)\ =\ \big(\partial_t\sqrt{J(t,x)}\big)\,a_0(t,\Phi(t,x))\ +\ \sqrt{J(t,x)}\,\Big(\partial_t a_0+\mathbf{v}.\partial_x a_0\Big)\big(t,\Phi(t,x)\big)\,.$$

Using the formula for $\partial_t\sqrt{J(t,x)}$, yields

$$\partial_t\Big(a_0(t,\Phi(t,x))\,\sqrt{J(t,x)}\Big)\ =\ \sqrt{J(t,x)}\,\Big(\partial_t a_0+\mathbf{v}.\partial_x a_0+\frac{1}{2}\big(\mathrm{div}\,\mathbf{v}\big)a_0\Big)\big(t,\Phi(t,x)\big)\,.$$

This completes the proof of the lemma. ∎

In the formula (2.23) take $y=\Phi(t,x)$ and use $a_0(t,y)=a_0(0,x)/\sqrt{J(t,x)}$ to find

$$\int |a_0(t,y)|^p\ dy\ =\ \int |a_0(0,x)|^p\ J(t,x)^{-p/2}\ dy\,.$$

Make the change of variable $y = \Phi(t,x)$, with $dy = J(t,x)\,dx$ to find,

$$\int |a_0(t,y)|^p \ dy \ = \ \int |a_0(0,x)|^p\, J(t,x)^{-p/2}\, J(t,x)\ dx\,.$$

For $1 \le p < 2$ using (2.6) yields the lower bound

$$\int |a_0(t,y)|^p \ dy \ \ge \ \Big(\frac{(1+t)^m}{2^N}\Big)^{(2-p)/2} \int |a_0(0,x)|^p \ dx\,. \tag{2.24}$$

Apply this formula with $t = T_\rho := \gamma\rho^{-1}$ to find

$$\lim_{\epsilon\to 0}\ \frac{\|u^\epsilon(T_\rho)\|_{L^p(\mathbb{R}^d)}}{\|u^\epsilon(0)\|_{L^p(\mathbb{R}^d)}} \ \ge \ \Big(\frac{(1+T_\rho)^m}{2^N}\Big)^{(2-p)/2p}\,.$$

Choosing ρ small, and therefore T_ρ large, this lower bound shows that the family of operators $e^{-itA(\partial)}$ is not uniformly bounded. However, if $e^{-itA(\partial)}$ were bounded on L^p for some $t \ne 0$ it would be bounded for all such t with norm independent of t. This contradiction shows that **i** is violated and completes the proof. ∎

§3. Proof ii ⇒ iii.

For real $\xi \in \mathbb{R}^d$, the matrix $A(\xi)$ has eigenvalues $\mathbf{v}_\mu.\xi$, $1 \le \mu \le \kappa$. Choose a $\underline{\xi}$ for which these values are distinct and denote by m_μ the multiplicity of the eigenvalue $\mathbf{v}_\mu.\underline{\xi}$. Then in a real neighborhood of $\underline{\xi}$ the eigenvalues remain distinct with the same multiplicities so one has

$$\det\big(\tau + A(\xi)\big) \ = \ \Pi_{\mu=1}^{\kappa}\big(\tau + \mathbf{v}_\mu.\xi\big)^{m_\mu}\,. \tag{3.1}$$

By analytic continuation this identity remains true for all $\xi \in \mathbb{C}^d$

Eigenvalue crossings occur exactly at those ξ so that

$$\exists\ \alpha \ne \beta\,, \quad (\mathbf{v}_\alpha - \mathbf{v}_\beta).\xi \ = \ 0\,. \tag{3.2}$$

Away from this resonance set, the eigenvalues are of locally constant multiplicity. The holomorphic spectral projections are defined by

$$\pi_\mu(\xi) \ := \ \frac{1}{2\pi i}\int_{|z-\mathbf{v}_\mu.\xi| = \rho << 1} \big(zI - A(\xi)\big)^{-1}\ dz\,. \tag{3.3}$$

They satisfy

$$\pi_\mu\,\pi_\nu = \delta_{\mu\nu}\,\pi_\mu\,, \qquad \sum_\mu \pi_\mu(\xi) = I\,,$$

and yield the spectral decomposition

$$A(\xi) \ = \ \sum_\mu\,(\mathbf{v}_\mu.\xi)\ \pi_\mu(\xi)\,. \tag{3.4}$$

The Fourier multiplier $e^{-it\mathbf{v}_\mu.\xi}$ yields the operator $e^{-it\mathbf{v}_\mu.\partial}$ which is equal to translation by $t\,\mathbf{v}_\mu$. This is an L^p isometry for all p.

The spectral decomposition shows that the Fourier multiplier $e^{-itA(\partial)}$ is equal to

$$\sum_\mu \pi_\mu(\partial)\, e^{-it\mathbf{v}_\mu.\partial}\, \pi_\mu(\partial) \;=\; \sum_\mu e^{-it\mathbf{v}_\mu.\partial}\, \pi_\mu(\partial)\,.$$

A rescaling in x shows that the norm of $e^{-itA(\partial)}$ on L^p is independent of t. Thus when this norm is finite one has

$$\Big\| \sum_\mu e^{-it\mathbf{v}_\mu.\partial}\, \pi_\mu(\partial) \Big\|_{\mathrm{Hom}(L^p)} \;\le\; C_p \qquad \text{independent of } t\,.$$

Since the $\mathbf{v}_\mu$ are distinct, when $t \to \infty$ the supports of $e^{-it\mathbf{v}_\mu.\partial} g$ separate and one finds that for $f \in L^p(\mathbb{R}^d)$ with $\|f\|_{L^p} = 1$,

$$\sum_\mu \|\pi_\mu(\partial) f\|_{L^p} \;=\; \lim_{t\to\infty} \Big\| \sum_\mu e^{-it\partial} \pi_\mu(\partial) f \Big\|_{L^p} \;\le\; C_p\,.$$

It follows that

$$\forall\, \mu\,, \quad \big\|\pi_\mu(\partial)\big\|_{\mathrm{Hom}(L^p)} \;\le\; C_p\,. \tag{3.5}$$

For $p = 2$, (3.5) implies that

$$\forall\, \mu, \quad \forall\, \xi \in \mathbb{R}^d, \qquad \big\|\pi_\mu(\xi)\big\|_{\mathrm{Hom}(\mathbb{C}^N)} \;\le\; C_2\,. \tag{3.6}$$

Changing coordinates in $\mathbb{R}^d$ we may assume that A_1 has κ distinct eigenvalues. The original coefficient matrices are linear combinations of the new coefficients so it suffices to show that the new coefficients are simultaneously diagonalisable.

Denote by $\underline{\pi}_\mu$, , $1 \le \mu \le \kappa$, the spectral projections of A_1. To prove the theorem it suffices to show that

$$\forall\, 1 \le \mu \le \kappa, \quad \forall\, 2 \le j \le d, \quad \exists \sigma_{j\mu} \in \mathbb{R}, \qquad A_j \underline{\pi}_\mu \;=\; \sigma_{j\mu}\, \underline{\pi}_\mu\,. \tag{3.7}$$

The proof of (3.7) is done for each j separately, ignoring all but A_1 and A_j. Thus it suffices to consider the case of two matrices, and relabeling if necessary the case $j = 2$.

In the complement of the set of resonant $\xi \in \mathbb{C}^2$ defined by (3.2), the projectors $\pi_\mu(\xi_1, \xi_2)$ are holomorphic functions on $\mathbb{C}^2$ homogeneous of degree 0 in the sense that for all $z \in \mathbb{C} \setminus 0 := \mathbb{C}^*$

$$\pi_\mu(z\xi_1, z\xi_2) \;=\; \pi_\mu(\xi_1, \xi_2)\,.$$

Each resonance relation is homogeneous and represents a single exceptional point in the complex projective space $\mathbb{C}\mathbf{P}^1 = (\mathbb{C}\times\mathbb{C})/\mathbb{C}^*$. Therefore the exceptional set is a finite set of points $p_1, \dots, p_M$ in $\mathbb{C}\mathbf{P}^1$ and π_μ defines a holomorphic function on $\mathbb{C}\mathbf{P}^1 \setminus \{p_1, \dots, p_M\}$

Since A_1 has κ distinct eigenvalues, the point $\xi = (1, 0)$ does not satisfy the resonance relation (3.2). Therefore, the singular points have homogeneous coordinates with $\xi_2 \neq 0$.

Lemma. *For all k and μ, the singularity of π_μ at p_k is removable.*

Proof of Lemma. From (3.6) we know that the π_μ are bounded at the real points.

Each singular point is given by a homogeneous equation $(\mathbf{v}_\alpha - \mathbf{v}_\beta).\xi = 0$ for some $\alpha \neq \beta$. The equation has the form

$$a_1\,\xi_1 + a_2\,\xi_2 \;=\; 0\,.$$

Since there are no solutions with $\xi_2 = 0$, it follows that $a_1 \neq 0$. The exceptional point is the line $\mathbb{C}^*(-a_2/a_1, 1) = C^*(\underline{z}, 1)$. The real points $\mathbb{R}^*(\epsilon + \underline{z}, 1)$ are not singular and approach the singular point. This together with (3.6) shows that there is a sequence of points approaching p along which π_μ is bounded. Therefore the singularity cannot be pole.

Near p in $(\mathbb{C}\times\mathbb{C})/\mathbb{C}^*$ use the homogeneous coordinates $(z,1)$ with $z \approx \underline{z}$.

The explicit linear formulas for the eigenvalues show that near $\underline{z}$, there is a constant $C_0 > 0$ so that for all $\alpha \neq \beta$ and z near $\underline{z}$,

$$|\lambda_\alpha(z,1) - \lambda_\beta(\underline{z},1)| \;\geq\; C_0|z-\underline{z}|\,. \tag{3.8}$$

Express $\pi_\mu(z,1)$ as a contour integral over the boundary of a disk of radius $C_0|z-\underline{z}|/2$

$$\pi_\mu(z,1) \;=\; \frac{1}{2\pi i}\oint_{|\zeta - \lambda_\mu(z,1)| = C_0|z-\underline{z}|/2} \Big(\zeta - (zA_1 + A_2)\Big)^{-1}\, d\zeta\,. \tag{3.9}$$

Estimate the determinant using (3.1), and (3.8) to find

$$\det\Big(\zeta - (A_1 + zA_2)\Big) \;\geq\; C_1|z-\underline{z}|^N\,, \qquad \text{with} \quad C_1 > 0\,.$$

Express the inverse in the contour integral (3.9) using Kramer's rule to find

$$\Big\|\Big(\zeta - (A_1 + zA_2)\Big)^{-1}\Big\| \;\leq\; C_2|z-\underline{z}|^{-N}\,.$$

Use this estimate in (3.8) to find

$$\big\|\pi_\mu(z,1)\big\| \;\leq\; C_3\,|z-\underline{z}|^{-N}\,.$$

It follows that π_μ cannot have an essential singularity.

The only remaining possibility is a removable singularity. ∎

Since the singularities are removable, each π_μ is entire on $\mathbb{C}\mathbf{P}^1$. Liouville's Theorem implies that the π_μ are constant.

The identities

$$\Big(A_1\xi_1 + A_2\xi_2\Big)\pi_\mu \;=\; (\mathbf{v}_\mu.\xi)\,\pi_\mu\,, \qquad 1 \leq \mu \leq \kappa$$

are known to hold outside the singular set. They therefore hold for all $\xi \in \mathbb{C}^2$ by analytic continuation. In particular for $\xi = (0,1)$ one finds

$$A_2\pi_\mu \;=\; \big(\mathbf{v}_\mu.(1,0)\big)\;\pi_\mu\,.$$

This is exactly the desired case $j = 2$ of formula (3.7), and the proof of **iii** is complete. ∎

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