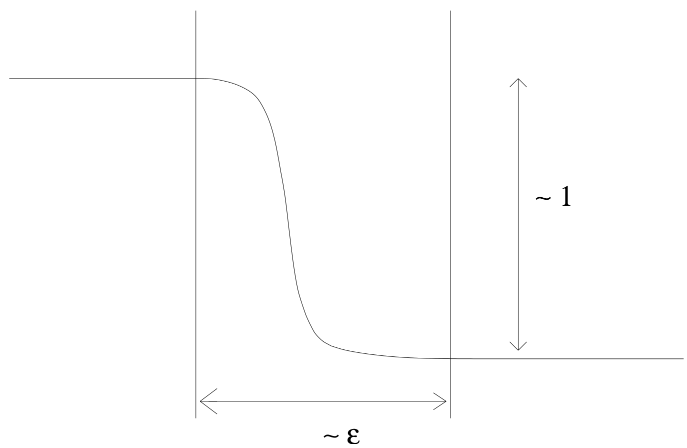
Nonlinear Asymptotics for Hyperbolic Internal Waves of Small Width

Olivier GUES
Université de Provence, Marseille, France
gues@cmi.univ-mrs.fr

Jeffrey RAUCH*
University of Michigan, Ann Arbor MI, USA
rauch@umich.edu

§1. Introduction.

In this paper we construct accurate approximations to solutions of hyperbolic partial differential equations which possess internal waves with thickness ε. The analysis is asymptotic as $\varepsilon \to 0$.



In the limit $\varepsilon \to 0$, the solutions converge to piecewise smooth functions which are discontinuous accross a characteristic surface. Such solutions have source terms which are also piecewise smooth. Discontinuous sources are idealizations of smooth sources with a thin ($\sim \varepsilon$) transition layer. The fundamental problem addressed here is to describe the dynamics of solutions with sources with such transition layers. The solutions have internal transition layers of size $\sim \varepsilon$. The limiting solution has a conormal singularity along the characteristic hypersurface. Our analysis employs conormal and $\varepsilon\partial$ estimates. The technical difficulty is that the obvious *ansatz* motivated by the cases of wave trains and short pulses yields overdetermined equations for correctors to the leading approximation. This is so even in the linear case. If one does not choose specially adapted coordinates, the transport equations differ from those describing the propagation of singularities and oscillations.

In a sense, the research is a sequel to the analysis of short pulses in [AR]. Pulses are internal waves with equal values on both sides of the wave. In the figure above the internal wave connects a higher value nearby to the left to a lower value nearby on the right. A pulse connects equal values. For short pulses, the obvious *ansatz* yields the correct leading term and a prescription for a first corrector which is overdetermined. [AR] relaxed the constraints on the first corrector and were able to prove that the leading term is an approximation with error $O(\varepsilon)$. They were unable to find higher order approximations.

We will show in §2, that for internal waves the most obvious *ansatz* again yields the correct recipe for the leading term and again an overdetermined first corrector. For internal waves we were not able

* Partially supported by the US National Science Foundation grant NSF-DMS-0104096

to simply relax the constraints to find a useful corrector. By adopting a different *ansatz* we are able to construct correctors of all orders and thereby infinitely accurate approximations. This advance at the level of formal asymptotics is a key innovation. The technique that we adopt is to consider both the limit problem and the smoothed problems as transmission problems. This strategy has been quite successful in the analysis of the inviscid limit of viscous shock waves ([GMWZ]), and of the viscous approach of discontinuous solutions of semilinear hyperbolic systems ([S]).

Consider a system of partial differential operators

$$L(t,x,\partial) \;=\; \partial_t + \sum A_j(t,x)\,\partial_j + B(t,x)\,, \qquad \partial_j \;:=\; \frac{\partial}{\partial x_j}\,, \tag{1.1}$$

where the A_j, B are infinitely differentiable $N\times N$ complex matrix valued functions each of whose partial derivatives is uniformly bounded on $\mathbb{R}\times\mathbb{R}^d$. Assume that L is hyperbolic in the following sense.

Assumption 1. *The system L is strictly hyperbolic, or symmetric hyperbolic.*

Recall that *strictly hyperbolic* means that the matrix $\sum_1^d \xi_j A_j$ has N simple eigenvalues

$$\lambda_1(t,x,\xi) < \ldots < \lambda_N(t,x,\xi)$$

for all $(t,x,\xi)\in\mathbb{R}\times\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})$. *Symmetric hyperbolic* means that the matrices A_j are hermitian symmetric (introduced in [F]).

We study an internal wave carried by a smooth characteristic hypersurface Σ of L. Assumption 1 is invariant under change of coordinates (t,x), hence, without loss a generality, we assume that we have chosen a set of local coordinates for which Σ is the set $\{x_d=0\}$. We consider only this local problem leaving to the interested reader questions of gluing local expansions together relying on the important Propositon 2.3. The prinicipal symbol is $L_1(t,x,\tau,\xi) := \tau I + \sum A_j(t,x)\,\xi_j$. The characteristic variety is $Char(L) := \big\{\det L_1(t,x,\tau,\xi)=0\big\}$.

Assumption 2. *$\Sigma := \{x_d = 0\}$ is a characteristic hypersurface for L. On a conic neighborhood of the conormal variety $\{(t,x',0;0,\ldots,0,\xi_d\neq 0)\}$ to Σ, the characteristic variety, Char(L), is a smooth embedded hypersurface $\tau=\tau(t,x,\xi)$ in the cotangent bundle of points $(t,x;\tau,\xi)\in\mathbb{R}^{1+d}_{t,x}\times\mathbb{R}^{1+d}_{\tau,\xi}$.*

Examples. i. Assumption 2 is always satisfied in the strictly hyperbolic case. The smoothness of the characteristic variety following from the implicit function theorem applied to the equation $\det L(\tau,\xi)=0$ for τ. The necessary hypothesis $\partial_\tau \det L_1|_{(\tau,\xi)=(0,\ldots,0,1)}\neq 0$ is implied by the simplicity of the roots.

ii. For a symmetric hyperbolic operator with constant coefficients, the characteristic variety is a real algebraic hypersurface in $\mathbb{R}^{1+d}$. The set of points where Assumtion 2 is violated is a subvariety of dimension not larger than $d-1$ so Assumption 2 is generic in the constant coefficient case.

iii. The Maxwell equations and the linearized compressible Euler equations are examples of symmetric systems which are not strictly hyperbolic but whose characteristic varieties are everwhere smooth and of constant multilicity. Assumtion 2 is therefore always satisfied for them.

Assumption 1 and 2 imply that $\tau(t,x',0;0,\ldots,0,\pm1)=0$ and $\dim\ker L_1(t,x,\tau(t,x,\xi),\xi)$ is constant for (t,x,ξ) in a conic neighborhood of $\{x_d=0\}\times\{\xi=(0,0,\ldots,\pm1)\}$. In particular

$\dim\ker A_d(t,x',0)$ is constant on Σ. Denote this dimension by k. Assumptions 1 and 2 are invariant by a smooth linear change of unknown $\tilde u = M(t,x)u$. We can therefore assume without loss of generality that such a change has been performed so that

$$A_d(t,x',0) \;=\; \begin{pmatrix} 0_{k\times k} & 0_{k\times N-k} \\ 0_{N-k\times k} & \mathcal{A}(t,x') \end{pmatrix}, \qquad \det \mathcal{A}(t,x') \;\geq\; \delta > 0\,. \tag{1.2}$$

The derivatives of the function τ play a central role in our results. Define the group velocity computed at the conormal variety to $\{x_d = 0\}$,

$$\mathbf{v}(t,x') \;:=\; -\nabla_\xi \tau(t,x',x_d=0,\tau=0,\xi'=0,\xi_d=1)\,.$$

Since τ vanishes on the conormal to Σ and is homogeneous of degree 1, it follows that $\mathbf{v}$ is tangent to $\{x_d = 0\}$.

The principal algebraic lemma of geometric optics asserts that Assumptions 1 and 2 imply that the differential operator $\underline{\pi} L(t,x',0,\partial)\underline{\pi}$ is essentially a directional derivative. The algebraic lemma is a consequence of first order perturbation theory in the following form. An eigenvalue λ of a matrix A is *semisimple* when the kernel and range of $A-\lambda I$ are complementary subspaces.

Proposition 1.1. *Suppose that* $]a,b[\ni s \to A(s)$ *is a smooth family of complex matrices with an isolated smooth semisimple eigenvalue* $\lambda(s)$*. Denote by* $\pi(s)$ *the spectral projection onto the kernel of* $A(s)-\lambda(s)I$ *along its range. Then*

$$\pi(s)\,\frac{dA(s)}{ds}\,\pi(s) \;=\; \frac{d\lambda(s)}{ds}\,\pi(s)\,.$$

Proof. Differentiate the identity $(A-\lambda)\pi = 0$ with respect to s denoting d/ds with a $'$ to find

$$(A-\lambda)'\,\pi \;+\; (A-\lambda)\,\pi' \;=\; 0\,.$$

Mulitplying by π eliminates the second term to yield

$$\pi\;(A-\lambda)'\,\pi \;+\; =\; 0\,,$$

which is the desired result. ∎

Denote by $\underline{\pi}$ the spectral projection of $L_1\big(t,x',0,(0,\dots,0,1)\big) = A_d(t,x')$ onto its kernel,

$$\underline{\pi}(u_1,\dots,u_d) \;=\; (u_1,\dots,u_k,0,\dots,0)\,.$$

Proposition 1.1 applied with $A(s) = A_d + sA_j$ and $\lambda(s) = -\tau(t,x',0;0,\dots,s,\dots,1)$ with small s in the ξ_j slot implies that

$$\underline{\pi}\;A_j\;\underline{\pi} \;=\; \mathbf{v}_j \qquad 1\leq j\leq d\,.$$

In particular, the classical transport operator of geometric optics satisfies

$$\underline{\pi}\;L(t,x',0,\partial)\;\underline{\pi} \;=\; \underline{\pi}\big(\partial_t + \mathbf{v}(t,x').\partial'_x\big) \;+\; \text{lower order terms}\,. \tag{1.3}$$

Proposition 1.1 with $A(s) = A_d(t, x', s)$ and $\lambda(s) = -\tau(t, x', s; 0, \ldots, 0, 1)$ yields

$$\underline{\pi}\, \frac{\partial A_d(t, x', 0)}{\partial x_d}\, \underline{\pi} \;=\; -\underline{\pi}\, \frac{\partial \tau}{\partial x_d}(t, x', x_d; 0, \ldots, 0, 1)\,.$$

The transport operator for internal waves, $\mathbb{H}$, is

$$\mathbb{H}(t, x; \partial_{t,x'}, z\partial_z) \;:=\; \underline{\pi}\, L(t, x', 0, \partial_{t,x})\, \underline{\pi} + \underline{\pi}\, \frac{\partial A_d(t, x', 0)}{\partial x_d}\, \underline{\pi}\, z\partial_z\,.$$

Lemma 1.1 shows that that the principal part of $\mathbb{H}$ is a scalar vector field,

$$\mathbb{H}(t, x; \partial_{t,x'}, z\partial_z) \;=\; \underline{\pi}\Big(\partial_t + \mathbf{v}(t, x').\nabla_{x'} + \partial_d \tau(t, x', 0; 0, \ldots, 0, 1) z\partial_z\Big) + \text{lower order terms}\,.$$

We study semilinear equations whose nonlinear term is an infinitely differentiable (in the real sense) function $G : \mathbb{C}^N \to \mathbb{C}^N$ satisfying $G(0) = 0$.

Main Problem. *Describe the behavior of solutions u^ε to*

$$L\, u^\varepsilon + G(u^\varepsilon) \;=\; f^\varepsilon\,, \qquad u^\varepsilon = f^\varepsilon = 0 \quad \text{when} \quad t < 0\,. \tag{1.4}$$

where

$$f^\varepsilon \;=\; F(t, x, x_d/\varepsilon)\,,$$

with $F(t, x, z)$ smooth, compactly supported in x, with limits

$$\lim_{\pm z \to \infty} F(t, x, z) \;=\; \overline{F}^{\pm}(t, x) \tag{1.5}$$

rapidly achieved.

Define a discontinuous piecewise smooth function

$$\overline{f}(t, x) \;:=\; \overline{F}^{\pm}(t, x)\,, \quad \text{when} \quad \pm\, x_d > 0\,,$$

The source term, f^ε, is a family converging to $\overline{f}$ as $\varepsilon \to 0$. For $\varepsilon > 0$, the discontinuity of $\overline{f}$ is replaced by a smooth transition layer of thickness $O(\varepsilon)$.
Passing to the limit $\varepsilon \to 0$ yields the initial value problem

$$L\bar{U} + G(\overline{U}) \;=\; \bar{f}\,, \qquad \overline{U} = \bar{f} = 0 \quad \text{for} \quad t < 0\,. \tag{1.6}$$

It is known that this problem has a local in time solution, $\overline{U} \in L^\infty([0, T_1] \times \mathbb{R}^d)$, whose restrictions, to the half slabs $[0, T_1] \times \{\pm x_d > 0\}$ are smooth up to the boundary and compactly supported (see [RR1], [RR2], [M1]). Denote by $\overline{U}^{\pm}$ the restriction to $\pm x_d > 0$.
The u^ε are smooth and the limit $\overline{U}$ is discontinuous so that the convergence is not uniform. The problem which we solve is to find correction terms to add to $\overline{U}$ so that the solution u^ε is described with small error in sup norm.

Define in $\pm z \geq 0$

$$\widetilde{F}_0^{\pm}(t, x', z) \;:=\; F(t, x', x_d = 0, z) - \overline{F}^{\pm}(t, x', x_d = 0)\,. \tag{1.7}$$

Denote by $\mathcal{Z}$ the conormal derivation $\varphi(x_d)\partial_d$ where $\varphi \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ is a fixed increasing function such that $\varphi(t) = t$ for $-1 < t < 1$, and $\varphi(t) = \pm 2$ for $\pm t > 3$.

Main Theorem. *Define in* $\{\pm x_d \geq 0\} \times \{\pm z \geq 0\}$ *the principal profile*

$$U_0^\pm \ := \ \overline{U}^\pm(t,x) + \widetilde{U}_0^\pm(t,x',z)\,, \tag{1.8}$$

where $\widetilde{U}_0^\pm\big(t,x',z\big) \in H^\infty\big([0,T_2]\times\mathbb{R}^{d-1}\times\mathbb{R}\big)$ *is determined as the local solution of the following nonlinear hyperbolic problem (with* $0 < T_2 \leq T_1$*),*

$$(I - \underline{\pi})\widetilde{U}_0^\pm = 0\,,$$

$$\mathbb{H}(t,x';\partial_{t,x'}, z\partial_z)\widetilde{U}_0^\pm + \underline{\pi}\Big(G(\overline{U}_0^\pm|_{x_d=0} + \widetilde{U}_0^\pm) - G(\overline{U}_0^\pm)\Big) \ = \ \underline{\pi}\widetilde{F}_0^\pm\,, \tag{1.9}$$

$$\widetilde{U}_0^\pm\big|_{t<0} \ = \ 0\,. \tag{1.10}$$

Then $u^\varepsilon - U_0(t,x,x_d/\varepsilon) = O(\varepsilon)$ *in the sense that if* ε *is sufficiently small then* u^ε *exists on* $[0,T_2]$ *and*

$$\forall\beta, \qquad \Big\| \big(\partial_{t,x'}\,,\,\mathcal{Z}\,,\,\varepsilon\partial_d\big)^\beta \Big(u^\varepsilon - U_0(t,x,x_d/\varepsilon)\Big)\Big\|_{L^\infty([0,T_2]\times\mathbb{R}^d_\pm)} \ = \ O(\varepsilon)\,. \tag{1.11}$$

Remarks. i. The $z\partial_z$ term in the transport equation (1.9) is not present in the transport equations describing the propogation of wave trains in geometric optics nor the propagation of singularities. Proposition 3.4 shows that if coordinates are chosen so that the hyperplanes $x_d =$ const. are all characteristic then this term is not present.

ii. We construct approximations of accuracy $O(\varepsilon^\infty)$ in the next sections.

§2. The ansatz.

The most obvious choice for an *ansatz* for the approximate solutions fails and it is important to understand where it fails. It is too restrictive to describe the solution. Introducing a more permissive *ansatz* we can do the same for the source term and thereby arrive at a more general setting.

§2.1. The source.

The exact form taken for the smoothed source is not too important. In particular one can pose sources of very restrictive form. The danger is that assuming such a restrictive form for the response may not leave enough flexibility. That is exactly what happens for what appears to us to be the most obvious *ansatz*.

In geometric optics, the form for oscillatory functions oscillating with phase $\phi(t,x)$ is

$$f^\varepsilon \ = \ f^\varepsilon(t,x,\phi/\varepsilon)\,, \qquad f^\varepsilon(t,x) \ \sim \ \sum_{j=0}^\infty \varepsilon^j\, f_j(t,x,\theta)\,,$$

with smooth f_j periodic with respect to θ.

The natural internal wave analogue of this with the transition at the surface $\{x_d = 0\}$ is

$$f^\varepsilon \ = \ F^\varepsilon(t,x,x_d/\varepsilon) \tag{2.1.1}$$

where $F^\varepsilon(t,x,z)$ has an asymptotic expansion

$$F^\varepsilon(t,x,z) \ \sim\ \sum_{j=0}^\infty \varepsilon^j\ f_j(t,x,z)\,, \quad \text{where the limits} \quad \lim_{\pm z\to\infty} f_j(t,x,z) \ =\ \overline{f}_j^{\pm}(t,x) \qquad (2.1.2)$$

are rapidly achieved.

Though this is adequate for the source f^ε in the next section we show why taking an analogous *ansatz* for the response fails.

§2.2 The most obvious u^ε ansatz fails.

A natural choice for u^ε is to mirror the structure of f^ε seeking u^ε in the form

$$u^\varepsilon(t,x',x_d) \ \sim\ \sum_{j=0}^\infty \varepsilon^j\, U_j(t,x,x_d/\varepsilon)\,, \qquad \lim_{z\to\pm\infty} U_j(t,x,z) \ =\ \overline{U}_j^{\pm}(t,x). \qquad (2.2.1)$$

A computation familiar from ordinary geometric optics and simpler than the one performed in detail in the next section yields the following equations which determine the leading profile $U_0(t,x,z)$.

The limits $\overline{U}_0^{\pm}(t,x) := \lim_{\pm z\to\infty} U_0(t,x,z)$ glue together to form a function

$$\overline{U}_0(t,x) \ :=\ \overline{U}_0^{\pm}(t,x) \quad \text{when} \quad \pm\, x_d > 0$$

which must satisfy the the initial value problem (1.4), whose piecewise smooth solution was our point of departure. Thus $\overline{U}_0 = \overline{U}$.

Denote by $Q(t,x')$ the partial inverse defined by

$$\underline{\pi} Q = 0\,, \quad QA_d(t,x',0) \ =\ I - \underline{\pi}\,, \qquad \text{so} \qquad Q = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{A}(t,x')^{-1}\end{pmatrix}\,. \qquad (2.2.2)$$

The tilde part (called the inner part in matched asymptotics) $\widetilde{U}_0^{\pm}(t,x',0,z)$ is determined as the local solution of the transport initial value problem in (1.9), (1.10).

The crunch comes with the first corrector equation, and already appears in the simplest case of a constant coeffient linear 1-d problem. Indeed, suppose that $d=1$, $G=0$ and $L := \partial_t + A\partial_x$ where the matrix $A = A_d$ is constant symmetric. The problem (1.4) is then

$$\partial_t u^\varepsilon + A\partial_x u^\varepsilon = F(t,x,x/\varepsilon)\,, \qquad u^\varepsilon|_{t<0} = 0\,.$$

The corresponding WKB profile equations for the profiles U_0 and U_1 are

$$A\partial_z U_0 = 0\,,$$

(for the terms in ε^{-1}) and,

$$\partial_t U_0 + A\partial_x U_0 + A\partial_z U_1 = F,$$

(for the terms in ε^0). Now, suppose that U_0 and U_1 satisfy these two equations. Applying the operator $A\partial_z$ to the second equation, and using the relation $A\partial_z U_0 = 0$ yelds the equation

$$\partial_z^2(A^2 U_1) = \partial_z(AF),$$

which is clearly NOT satisfied in general, since an integration with respect to z would give the relation

$$0 = AF(t, x, +\infty) - AF(t, x, -\infty)$$

which is not true in general. Therefore a smooth corrector U_1 does not usually exist.

In the general case this obstruction persists. The equation for the ε^0 term in $\{x_d = 0\}$ is,

$$A_d(t, x', 0)\, \partial_z \widetilde{U}_1 \;=\; L(t, x', 0, \partial_{t,x})\widetilde{U}_0 + G(\overline{U}_0^{\pm}|_{x_d=0} + \widetilde{U}_0^{\pm}) - G(\overline{U}_0^{\pm}) - \widetilde{F}_0(t, x', z = 0)\,.$$

The equations for $\widetilde{U}_0$ guarantee that the right hand side is in the image of A_d so

$$(I - \underline{\pi})\partial_z \widetilde{U}_1 \;=\; Q\Big(L(t, x', 0, \partial_{t,x})\widetilde{U}_0 + G(\overline{U}_0^{\pm}|_{x_d=0} + \widetilde{U}_0^{\pm}) - G(\overline{U}_0^{\pm}) - \widetilde{F}_0\Big)$$

is determined and rapidly decreasing as $|z| \to \infty$. However, in order to guarantee that $(I - \underline{\pi})\big(\widetilde{U}_1(t, x, +\infty) - \widetilde{U}_1(t, x, -\infty)\big) = 0$ requires the moment condition

$$\int_{-\infty}^{\infty} L(t, x', 0, \partial_{t,x})\widetilde{U}_0(t, x, z) + G(\overline{U}_0^{\pm}|_{x_d=0} + \widetilde{U}_0^{\pm}) - G(\overline{U}_0^{\pm}) - \widetilde{F}_0(t, x', 0, , z)\; dz \;=\; 0$$

for all $(t, x', 0)$. This is generically violated, *even in the linear case.*

Similar difficulties with moment conditions occured in the work of Alterman-Rauch [AR] (see also [BL], [T]) on short pulses where the natural *ansatz* would have profiles which tend to zero as $|z| \to \infty$. These authors relaxed the requirement on the first corrector U_1 to allow U_1 to have nonvanishing limit at $z = +\infty$. The moment condition then created a crunch in determining U_2 but a first corrector worked. In the present context, the limit of U_0 at $z = \pm\infty$ are already unequal and the crunch occurs in the determination of U_1. We overcome this problem and thereby treat the internal layers and improve the results on pulses.

§**2.3. The transmission strategy.**

A hint that the moment condition should not be a fatal stumbling block comes from the following remark. In $U_0(t, x, z)$ one makes the substitution $z = x_d/\varepsilon$. Thus in $x_d > 0$ only the limit at $z = \infty$ counts and in $x_d < 0$ only the limit at $z = -\infty$ counts. One never really needs to have both $z = \pm\infty$ limits. To capitalize on this, it is natural to split the problem according to the two sides $\pm x_d > 0$. This corresponds to the transmission problem strategy which has been successful in the related problem of viscous perturbations of shocks [GMWZ], and of semilinear discontinuous waves [S]. In those results on boundary layers and shock structure, many tools have been borrowed from geometric optics. In this paper the favor is returned as we borrow from them to treat a problem of geometric optics.

The initial value problem (1.4) is equivalent to the transmission problem

$$L\, u^\varepsilon + G(u^\varepsilon) \;=\; f^\varepsilon \quad \text{in} \quad \{x_d \neq 0\}\,, \qquad \Big[(I - \underline{\pi})u^\varepsilon\Big]_{x_d=0} \;=\; 0\,, \tag{2.3.1}$$

where the square brackets indicate the jump from $(t, x', x_d = 0-)$ to $(t, x', x_d = 0+)$.

To advance we make weaker requirements on the approximate solution u^ε than in the preceding subsection. The *ansatz* for u^ε now has profiles for each half space. Begin with

$$u^\varepsilon \;=\; U^\varepsilon(t, x, x_d/\varepsilon) \tag{2.3.2}$$

where, $U^\varepsilon(t,x,z)$ is compactly supported in x with asymptotic expansions

$$U^\varepsilon(t,x,z) \ \sim \ \sum_{j=0}^{\infty} \varepsilon^j\, U_j^\pm(t,x,z)\,, \qquad \text{in} \quad \{\pm x_d \geq 0\} \times \{\pm z \geq 0\}\,, \tag{2.3.3}$$

$$U_j^\pm(t,x,z) \ = \ \overline{U}_j^\pm(t,x) + \widetilde{U}_j^\pm(t,x,z)\,,$$

with $\widetilde{U}_j^\pm$ rapidly decreasing as $\pm z \to \infty$. We do **not** require that $\widetilde{U}^\pm \to 0$ when $z \to \mp\infty$. In fact, $\widetilde{U}^\pm$ is not even defined at such points.

Because of the rapid decrease, $\widetilde{U}_j(t,x,x_d/\varepsilon)$ is essentially supported in an ε neighborhood of $x_d = 0$. More precisely, in defining u^ε one always has $x_d = \varepsilon z$ which suggests Taylor expansion in x_d,

$$\widetilde{U}_j^\pm(t,x',x_d,z) \ \sim \ \sum_{k=0}^{\infty} \frac{x_d^k}{k!}\, \partial_{x_d}^k \widetilde{U}_j^\pm(t,x',0,z)\,.$$

Replacing x_d by εz yields an equivalent profile with the property that the z dependent parts depend only on t, x', z and not on x_d. Note that the $x_d^k = \varepsilon^k z^k$ term appears as parts of the new profile at order $j+k$.

This leads to the final form for the *ansatz* (2.3.2)-(2.3.3) where

$$U_j^\pm(t,x,z) \ = \ \overline{U}_j^\pm(t,x) + \widetilde{U}_j^\pm(t,x',z) \tag{2.3.4}$$

with $\widetilde{U}_j^\pm$ independent of x_d and rapidly decreasing as $\pm z \to \infty$.

Precisely for U^ε defined on $t \leq T$ one requires that the support is contained in a compact subset of $[0,T] \times \mathbb{R}^d$ and for all α and $N \geq 0$

$$\sup_{\{0 \leq t \leq T\} \times \{\pm x_d \geq 0\} \times \{\pm z \geq 0\}} \left| \partial_{t,x,z}^\alpha \Big(U^\varepsilon(t,x,z) - \sum_{j=0}^{N} \varepsilon^j\, U_j^\pm(t,x,z) \Big) \right| \ = \ O(\varepsilon^{N+1}) \ \text{ as } \ \varepsilon \to 0+\,, \tag{2.3.5}$$

and

$$\sup_{[0,T] \times \{\pm x_d > 0\} \times \{\pm z > 0\}} \left| \langle z \rangle^N \partial_{t,x,z}^\alpha \Big(U_j^\pm(t,x',z) - \overline{U}_j^\pm(t,x) \Big) \right| \ < \ \infty\,, \quad \langle z \rangle := (1 + |z|^2)^{1/2}\,. \tag{2.3.6}$$

Proposition 2.1. *If a family of function u^ε has an asymptotic expansion of form (2.3.2)…(2.3.6), then the profiles $\overline{U}_j^\pm$ and $\widetilde{U}_j^\pm$ are uniquely determined.*

Proof. The leading barred terms in $\{\pm x_d > 0\}$ are given by

$$\overline{U}_0^\pm(t,x) \ = \ \lim_{\varepsilon \to 0^+} \ u^\varepsilon(t,x)\,.$$

The leading tilde term in $\pm z > 0$ is given by

$$\widetilde{U}_0^\pm(t,x',z) \ = \ \lim_{\varepsilon \to 0^+} \Big(u^\varepsilon(t,x',\varepsilon z) - \overline{U}_0^\pm(t,x',\varepsilon z) \Big)\,. \tag{2.3.7}$$

Inductively suppose $k \geq 1$ and $U_j^\pm$ are uniquely determined for $j \leq k-1$. The $U_k^\pm$ are given by

$$\overline{U}_k^\pm(t,x) \ = \ \lim_{\varepsilon\to 0^+} \ \varepsilon^{-k}\Big(u^\varepsilon(t,x) - \sum_0^{k-1} \varepsilon^j \, U_j^\pm(t,x,x_d/\varepsilon)\Big)\,,$$

$$\widetilde{U}_k^\pm(t,x',z) \ = \ \lim_{\varepsilon\to 0^+} \ \varepsilon^{-k}\bigg(u^\varepsilon(t,x',\varepsilon z) - \sum_0^{k-1} \varepsilon^j \, U_j^\pm(t,x',\varepsilon z,\varepsilon z) - \varepsilon^k\, \overline{U}_k^\pm(t,x',\varepsilon z)\bigg)\bigg)\,. \quad \blacksquare$$

By gluing define

$$\overline{U}_j(t,x) \ := \ \begin{cases} \overline{U}_j^+(t,x) & \text{when} \quad x_d > 0 \\ \overline{U}_j^-(t,x) & \text{when} \quad x_d < 0\,. \end{cases} \tag{2.3.8}$$

It costs us nothing to consider the more general problem with sources

$$f^\varepsilon \ = \ F^\varepsilon(t,x,x_d/\varepsilon) \tag{2.3.9}$$

with expansions like those of u^ε. That is, in $\{t \leq T_0 < \infty\} \times \{\pm x_d \geq 0\} \times \{z \geq 0\}$, $F^\varepsilon(t,x,z)$ has asymptotic expansion

$$F^\varepsilon(t,x,z) \ \sim \ \sum_{j=0}^\infty \varepsilon^j\, F_j^\pm(t,x,z)\,, \qquad F_j^\pm(t,x,z) = \overline{F}_j^\pm(t,x) + \widetilde{F}^\pm(t,x',z)\,, \tag{2.3.10}$$

with $F_j^\pm$ defined in $\pm x_d \geq 0$, $\widetilde{F}_j^\pm(t,x',z)$ defined in $\pm z \geq 0$, both compactly supported in x and with $\widetilde{F}_j^\pm$ rapidly decreasing as $\pm z \to +\infty$. The precise form is as in (2.3.2, …, 2.3.6).

Warning. The main problem concerns smoothed jumps. However, the relaxed expansions (2.3.10) suggested by the transmission strategy include functions f^ε which are discontinuous across $x_d = 0$. Functions f^ε and u^ε with expansions as above are always piecewise smooth. The source f^ε is continuous across $x_d = 0$ to leading order if and only if

$$\overline{f}_0^+(t,x',x_d=0) \ + \ \tilde{f}_0^+(t,x',z=0) \ = \ \overline{f}_0^-(t,x',x_d=0) \ + \ \tilde{f}_0^-(t,x',z=0)\,. \tag{2.3.11}$$

Equivalently

$$\Big[\,\overline{f}_0\,\Big]_{x_d=0} \ + \ \Big[\,\tilde{f}_0\,\Big]_{z=0} \ = \ 0\,. \tag{2.3.12}$$

In this case, the leading term $f_0(t,x,x_d/\varepsilon)$ in the source is a continuous transition layer of width ε which tends in the limit $\varepsilon \to 0$ to $\overline{f}$ which is the source term in (1.4). In Lemma 3.1 we will show that when f^ε is continuous to leading order, the same is true of the response u^ε. Corollary 6.2 gives a C^∞ analog.

We assume that

$$f \ = \ \overline{f}_j^\pm \ = \ \tilde{f}_j^\pm \ = \ 0\,, \qquad \text{when} \qquad t < 0\,. \tag{2.3.13}$$

There is a second and very different way to generate smoothed sources f^ε which is to take a standard mollification of the piecewise smooth source $\overline{f}$. This second is included in the sources (2.3.10) as the next Proposition shows.

Suppose that $j(t,x)$ is smooth compactly supported in $t \geq 0$ and with integral equal to one. Define $j^\varepsilon(t,x) = \varepsilon^{-d-1} j(t/\varepsilon, x/\varepsilon)$ and denote by J^ε the smoothing operator which is convolution with j^ε. Suppose that $\overline{f}$ is piecewise smooth and compactly supported on on $\{t \leq T\} \times \mathbb{R}^d$ with jumps on $\{x_d = 0\}$.

Proposition 2.2. *With the hypotheses of the preceding paragraph, $f^\varepsilon := J^\varepsilon \overline{f}$ has an asymptotic expansion of the form (2.3.10).*

Sketch of Proof. Define

$$\gamma(x_d) \ := \ \int j(t, x', x_d) \ dt \, dx' \, .$$

Denote by Γ^ε the operator which is convolution in x_d with $\varepsilon^{-1}\gamma(x_d/\varepsilon)$. The smoothness of $\overline{f}$ with respect to t, x' implies that the restriction of $\Gamma^\varepsilon \overline{f} - J^\varepsilon \overline{f}$ to each half space $\{\pm x_d \geq 0\}$ has compactly supported partial derivates of size $O(\varepsilon^\infty)$. Thus it suffices to show that $\Gamma^\varepsilon \overline{f}$ has an expansion.

This reduces to a problem in one dimension on $\mathbb{R}_{x_d}$ with t, x' as parameters. Consider the one dimension problem with scalar variable x.

Write $\overline{f}$ as the sum of a smooth function and a piecewise smooth function compactly supported in $x \geq 0$. Γ^ε applied to the smooth part is equal to the smooth part plus $O(\varepsilon^\infty)$. This part of $\Gamma^\varepsilon \overline{f}$ has an expansion (2.3.2)-(2.3.3) without tilde terms and with a single barred term equal to the smooth part. This reduces to the case of $\overline{f}$ with compact support in $x \geq 0$.

If $\overline{f}$ were infinitely flat at $x = 0+$, then $\overline{f}$ would be smooth and the difference $\Gamma^\varepsilon \overline{f} - \overline{f}$ would be $O(\varepsilon^\infty)$ so that there would be an expansion (2.3.2-3) with the single term $\overline{f}(x)$ on the right.

Taylor's Theorem expresses

$$\overline{f} \ \sim \ \sum_{j=0}^{\infty} \frac{x_+^k}{k!} \, \partial_x^k \overline{f}(0+) \ + \ O(x_+^\infty) \, .$$

We have just remarked that the $O(x_+^\infty)$ term is OK.

For the Taylor term, an exact evaluation yields

$$\Gamma^\varepsilon(x_+^k) = \gamma^\varepsilon * x_+^k = \varepsilon^k q_k(x/\varepsilon) \, , \qquad q_k(x) := \gamma * (x_+^k) \, .$$

Then

$$\Gamma^\varepsilon \overline{f} \ \sim \ \sum_{j=0}^{\infty} \frac{\varepsilon^k q_k(x/\varepsilon)}{k!} \, \partial_x^k \overline{f}(0+) \ + \ O(\varepsilon^\infty) \, ,$$

which is an expansion of type (2.3.2-3) containing only layer terms.

To fill in the details one takes this argument with a finite Taylor expansion with remainder to treat $x < 1$. The set $x > 1/2$ poses no problem. A two member partition of unity suffices to cover $x \geq 0$. This completes the sketch of proof. ∎

Proposition 2.3. *The set of families u^ε which have expansions of the form (2.3.2)-(2.3.6) is invariant under smooth change of coordinates*

$$\big(\tilde{t}, \tilde{x}\big) = \big(\tilde{t}(t,x), \tilde{x}(t,x)\big) \, , \qquad (t,x) = \big(t(\tilde{t}, \tilde{x}), x(\tilde{t}, \tilde{x})\big)$$

which map the half spaces $\pm x_d > 0$ to the corresponding halfspaces $\pm \tilde{x}_d > 0$.

Remarks. 1. This result is local in x and we suppose in the next calculations that the families are compactly supported within the domain where the change of variables is defined. **2.** An important special case occurs if one makes a change of defining function of Σ. If $\phi(t,x)$ has nondegenerate zero exactly at $x_d = 0$, then families of the form $u^\varepsilon = U(t,x,\phi(t,x)/\varepsilon)$ with $U_j(t,x,z)$ as before are the same as those with defining function x_d.

Sketch of proof. Denote

$$y = (y_0, y_1, \dots, y_d) = (t,x)\,, \qquad y\prime = (y_0, y_1, \dots, y_{d-1})\,,$$

$$\tilde y = (\tilde y_0, \tilde y_1, \dots, \tilde y_d) = (\tilde t, \tilde x), \qquad \tilde y' = (\tilde y_0, \tilde y_1, \dots, \tilde y_{d-1})\,.$$

Suppose that (2.3.2)-(2.3.6) hold. In the new variables we have

$$u^\varepsilon(\tilde y) \;\sim\; \sum \varepsilon^j \, \Big(\overline{U}_j^{\pm}(y(\tilde y)) + \widetilde{U}_j^{\pm}(y'(\tilde y), y_d(\tilde y)/\varepsilon) \Big)\,.$$

This is not of the desired form because in the layer term one has y_d/ε and not $\tilde y_d/\varepsilon$ and the tilde profile has y_d dependence on the slow scale.

Since the halfspaces are preserved, one has

$$y_d(\tilde y) = a(\tilde y)\, y_d$$

with smooth a. Therefore,

$$\widetilde{U}_j^{\pm}(y'(\tilde y), y_d(\tilde y)/\varepsilon) \;=\; \widetilde{U}_j^{\pm}\Big(y'(\tilde y), a(\tilde y)\frac{\tilde y_d}{\varepsilon}\Big) \;=\; \widetilde{V}_j\Big(\tilde y\,,\, \frac{\tilde y_d}{\varepsilon}\Big)$$

where

$$\widetilde{V}_j^{\pm}(\tilde y, z) \;:=\; \widetilde{U}_j^{\pm}\Big(y'(\tilde y)\,,\, a(\tilde y) z\Big)\,.$$

Replacing $\widetilde{V}_j^{\pm}$ by its Taylor expansion

$$\widetilde{V}_j^{\pm} \;\sim\; \sum_{k=0}^{\infty} \frac{\tilde y_d^k}{k!}\, \frac{\partial^k \widetilde{V}_j^{\pm}(\tilde y_0, \dots, \tilde y_{d-1}, 0, z)}{\partial \tilde y_d^k} \;\sim\; \sum_{k=0}^{\infty} \frac{\varepsilon^k\, z^k}{k!}\, \frac{\partial^k \widetilde{V}_j^{\pm}(\tilde t, \tilde x', 0, z)}{\partial \tilde y_d^k}\,,$$

yields an expansion of the desired form. ∎

§3. The profile equations.

Having settled on the *ansatz* for u^ε, the computation of the equations determining the profiles is a bit tricky but follows standard practice.

The transmission condition from (2.3.1) is satisfied to $O(\varepsilon^\infty)$ if and only if for all j, t, x',

$$\begin{aligned} 0 =& (I - \underline{\pi})\Big(U_j^+(t,x',x_d = 0+, z = 0+) - U_j^-(t,x',x_d = 0-, z = 0-) \Big) \\ =& (I - \underline{\pi})\Big(\overline{U}_j^+(t,x',x_d = 0+) + \widetilde{U}_j^+(t,x',z = 0+) - \overline{U}_j^-(t,x',x_d = 0-) - \widetilde{U}_j^-(t,x',z = 0-) \Big)\,. \end{aligned} \tag{3.1}$$

When these conditions are satisfied one can choose $U^\varepsilon(t,x,z)$ so that the transmission condition is exactly satisfied.

Proposition 3.1. *If u^ε and f^ε have expansions of the form (2.3.3)-(2.3.6) and u^ε satisfies the transmission condition exactly then $Lu^\varepsilon + G(u^\varepsilon) - f^\varepsilon$ has an expansion*

$$Lu^\varepsilon + G(u^\varepsilon) - f^\varepsilon \;=\; W^\varepsilon(t,x,x_d/\varepsilon) \;\sim\; \sum_{j=-1}^{\infty} \varepsilon^j \; W_j(t,x,x_d/\varepsilon) \tag{3.2}$$

where W_j is compactly supported in x and smooth in $\pm x_d \geq 0, \pm z \geq 0$, and

$$W_j(t,x,z) \;=\; \overline{W}_j^{\pm}(t,x) + \widetilde{W}_j^{\pm}(t,x',z) \tag{3.3}$$

with $\widetilde{W}_j^{\pm}(t,x',z)$ rapidly decreasing as $\pm z \to \infty$.

Note that the leading term in the expansion is in ε^{-1} in contrast to the expansion of $u^\varepsilon, f^\varepsilon$ which start at ε^0.

Proof. Thanks to the transmission condition, there are no delta functions when one computes $L\,u^\varepsilon$. Computing $L\,u^\varepsilon$ in $\pm x_d > 0$, one finds

$$L\,u^\varepsilon \;=\; \ell(\varepsilon,t,x,x_d/\varepsilon)\,, \qquad \ell(\varepsilon,t,x,z) = L\Big(t,x,\partial_t,\partial_x + \frac{1}{\varepsilon}\partial_z\Big)U^\varepsilon(t,x,z)\,. \tag{3.4}$$

This yields an expansion of the desired type for Lu^ε as follows. Letting $z \to \infty$ yields the barred part

$$L\Big(t,x,\partial_t,\partial_x\Big)\overline{U}^\varepsilon(t,x)\,.$$

The tilde part comes from the difference which is equal to

$$L\Big(t,x,\partial_t,\partial_x + \frac{1}{\varepsilon}\partial_z\Big)\widetilde{U}^\varepsilon(t,x',z)\,.$$

Taylor expansion yields an equivalent (modulo ε^∞) operator when $x_d = \varepsilon z$ with coefficients depending only on t, x'

$$\tilde{L}(t,x',\partial_{t,x}) \;:=\; \partial_t + \sum_{j=0}^{d}\sum_{k=0}^{\infty}\left(\frac{\partial A_j(t,x',0)}{\partial x_d^k}\;\frac{\varepsilon^k\, z^k}{k!}\right)\frac{\partial}{\partial x_j}\,.$$

The tilde part of Lu^ε has expansion

$$\tilde{L}\Big(t,x',\partial_t,\partial_x + \frac{1}{\varepsilon}\partial_z\Big)\;\sum_j \varepsilon^j\,\widetilde{U}_j^{\varepsilon,\pm}(t,x',z)\,.$$

The treatment of the nonlinear term $G\big(U^\varepsilon(t,x,z)|_{x_d=\varepsilon z}\big)$ is by Taylor expansion yielding

$$G(U^\varepsilon(t,x,z)) \;\sim\; \sum_{j=0}^{\infty}\varepsilon^j\; G_j^{\pm}(t,x,z)\,, \qquad G_j^{\pm}(t,x,z) = \overline{G}_j^{\pm}(t,x) + \widetilde{G}_j(t,x',z)\,. \tag{3.5}$$

The leading term $G_0^{\pm}$ comes from $G\big(U_0^{\pm}(t,x,z)\big)$. The limit $z \to \pm\infty$ yields the barred part

$$\overline{G}_0^{\pm}(t,x) = G(\overline{U}_0^{\pm})\,.$$

The tilde part comes from the difference $G(\overline{U}_0^\pm + \widetilde{U}_0^\pm) - G(\overline{U}_0^\pm)$. Taylor expansion in x_d yields the equivalent expression

$$G\Big(\sum_{k=0}^{\infty} \frac{\partial^k \overline{U}_0^\pm(t,x',0)}{\partial x_d^k}\,\frac{\varepsilon^k\,z^k}{k!} + \widetilde{U}_0^\pm\Big) \;-\; G\Big(\sum_{k=0}^{\infty} \frac{\partial^k \overline{U}_0^\pm(t,x',0)}{\partial x_d^k}\,\frac{\varepsilon^k\,z^k}{k!}\Big).$$

In this expression one performs a Taylor expansion about the ε^0 terms. Passing higher order terms in ε to the W_j with $j \geq 1$ yields the ε^0 terms

$$G(\overline{U}_0^\pm|_{x_d=0} + \widetilde{U}_0^\pm) - G(\overline{U}_0^\pm|_{x_d=0})\,.$$

An alternate expression uses Taylor's theorem

$$G(U+V) - G(U) = G_1(U,V)\,V\,, \qquad G_1(U,V) := \int_0^1 G'(U+sV)\,ds\,, \tag{3.6}$$

so

$$G(\overline{U}_0^\pm|_{x_d=0} + \widetilde{U}_0^\pm) - G(\overline{U}_0^\pm) \;=\; G_1(\overline{U}_0^\pm|_{x_d=0}, \widetilde{U}_0^\pm)\,\widetilde{U}_0^\pm\,. \tag{3.7}$$

For the terms of order $j \geq 1$, one has

$$\overline{G}_j^\pm(t,x) = G'(\overline{U}_0^\pm)\,\overline{F}_j^\pm + H(\overline{U}_0^\pm, \dots, \overline{U}_{j-1}^\pm)$$

$$\widetilde{G}_j^\pm(t,x',z) = G'(\overline{U}_0^\pm|_{x_d=0})\,\widetilde{F}_j^\pm + K(U_0, \dots, U_{j-1})$$

where the H, K terms are a smooth function of the earlier profiles. This structure adapts well to a recursive determination of the $U_j^\pm$. Note that in performing this computation, when one encounters a product of a barred term and a tilde term, the bar term is replaced by its Taylor expansion at $x_d = 0$ in order to give tilde terms which depend only on t, x', z.

Combining the above expressions for Lu^ε, $G(u^\varepsilon)$ with the expansion for f^ε completes the proof. ∎

The computation of the terms W_j in the above algorithm is straight forward, but the formulas get complicated.

The $j = -1$ term comes from

$$A_d(t,x',x_d)\,\partial_z \widetilde{U}_0^\pm(t,x',z) \;=\; \Big(A_d(t,x',0) + \big(A_d(t,x',\varepsilon z) - A_d(t,x',0)\big)\Big)\partial_z \widetilde{U}_0^\pm(t,x',z)\,.$$

The first summand yields the $j = -1$ term.

$$W_{-1} \;=\; \widetilde{W}_{-1} \;=\; A_d(t,x',0)\,\partial_z \widetilde{U}_0^\pm(t,x',z)\,. \tag{3.8}$$

The $j = 0$ term comes from

$$L(t,x,\partial_{t,x})U_0^\pm(t,x) + A_d(t,x)\,\partial_z U_1^\pm + G(U_0^\pm) + \varepsilon^{-1}\big(A_d(t,x',\varepsilon z) - A_d(t,x',0)\big)\partial_z \widetilde{U}_0^\pm(t,x',z) - F_0^\pm.$$

Letting $z \to \infty$ yields the bar part of this equation

$$\overline{W}_0 \;=\; L(t,x,\partial_{t,x})\overline{U}_0^\pm(t,x) + G(\overline{U}_0^\pm) - \overline{F}_0^\pm\,. \tag{3.9}$$

The tilde part is not as fast since we must extract the z depend profile which does not depend on x_d. For the first two terms that comes from Taylor expansion to yield

$$L(t,x',0,\partial_{t,x})\,\widetilde{U}_0^\pm(t,x',z) + A_d(t,x',0)\,\partial_z\widetilde{U}_1^\pm\,.$$

Writing

$$A_d(t,x',\varepsilon z) - A_d(t,x',0) \;=\; \varepsilon z\,\partial_d A_d(t,x',0) + O(\varepsilon^2)\,,$$

yields

$$\widetilde{W}_0 = L(t,x',0,\partial_{t,x})\widetilde{U}_0^\pm(t,x) + A_d(t,x',0)\,\partial_z\widetilde{U}_1^\pm + G_1(\overline{U}_0^\pm|_{x_d=0},\widetilde{U}_0^\pm)\widetilde{U}_0^\pm + z\partial_d A_d(t,x',0)\partial_z\widetilde{U}_0^\pm - \widetilde{F}_0^\pm\,. \tag{3.9}$$

The terms W_j with $j \geq 1$ are similar

$$\overline{W}_j^\pm \;=\; L(t,x,\partial_{t,x})\overline{U}_j^\pm(t,x) + G'(\overline{U}_0^\pm)\overline{U}_j^\pm - \overline{F}_j^\pm + H_j^\pm(\overline{U}_0^\pm,\ldots,\overline{U}_{j-1}^\pm)\,, \tag{3.10}$$

$$\begin{aligned}\widetilde{W}_j^\pm = L(t,x',0,\partial_{t,x})\widetilde{U}_j^\pm + G'(U_0^\pm|_{x_d=0})\widetilde{U}_j + A_d(t,x',0)\,\partial_z\widetilde{U}_{j+1}^\pm \\ + z\partial_d A_d(t,x',0)\partial_z\widetilde{U}_j^\pm - \widetilde{F}_j^\pm + K_j^\pm(U_0^\pm,\ldots,U_{j-1}^\pm)\,.\end{aligned} \tag{3.11}$$

The exact form of the H and K terms in not crucial. What is important is that they are determined by earlier profiles and are bar and tilde profiles respectively.

We construct $U_j^\pm$ in such a way that all the $W_j^\pm$ vanish identically.

The equation $W_{-1} = 0$ is equivalent to

$$(I - \underline{\pi})\,\partial_z\widetilde{U}_0^\pm(t,x',z) \;=\; 0\,.$$

Since $\widetilde{U}_0^\pm(t,x',\pm\infty) = 0$, this is equivalent to $\widetilde{U}_0$ satisfying the polarization identity

$$\underline{\pi}\,\widetilde{U}_0^\pm(t,x',z) \;=\; \widetilde{U}_0^\pm(t,x',z)\,. \tag{3.12}$$

This together with (3.1) implies the jump condition,

$$(I - \underline{\pi})\Big(\overline{U}_j^+(t,x',x_d=0+) - \overline{U}_j^-(t,x',x_d=0-)\Big) \;=\; 0\,. \tag{3.13}$$

Setting $\overline{W}_0 = 0$ yields

$$L\,\overline{U}_0^\pm + G'(\overline{U}_0^\pm) \;=\; \overline{F}_0^\pm\,. \tag{3.14}$$

This together with (3.13) shows that $\overline{U}_0^\pm$ must be the $\pm x_d > 0$ parts of the piecewise smooth solution from (1.4). Thus $\overline{U}_0$ is equal to the function $\overline{U}$ from (1.4).

Setting $\underline{\pi}\widetilde{W}_0^\pm = 0$ yields the nonlinear hyperbolic equation determining $\widetilde{U}_0$,

$$\underline{\pi}L(t,x',0,\partial_{t,x})\underline{\pi}\,\widetilde{U}_0^\pm + \underline{\pi}\,G_1(\overline{U}_0^\pm|_{x_d=0},\widetilde{U}_0^\pm)\widetilde{U}_0^\pm + \underline{\pi}\,z\partial_d A_d(t,x',0)\partial_z\widetilde{U}_0^\pm \;=\; \underline{\pi}\widetilde{F}_0^\pm\,, \tag{3.15}$$

$$\widetilde{U}_0^\pm = 0 \quad \text{when} \quad t < 0\,. \tag{3.16}$$

The operator $\mathbb{H}$ from the introductory section appears here. In particular, equation (3.15) is a transport equation along the integral curves of $\partial_t + \mathbf{v}.\partial_{x'} + \partial_d\tau\, z\partial_z$.

Formula (1.3) shows that (3.15) is a nonlinear transport equation with velocity parallel to $\{x_d = 0\}$. Therefore the initial value problem (3.15)-(3.16) uniquely determines $\widetilde{U}_0^\pm$ from $\underline{\pi}\widetilde{F}_0^\pm$. The rapid decay of $\widetilde{U}_0^\pm$ as $\pm z \to \infty$ follows from the corresponding decay of $\widetilde{F}_0^\pm$. The proof of the rapid decay is parallel to that of Proposition 2.3 of [G3].

For the next Proposition, recall that the leading term in the expansion of f^ε is continuous across $x_d = 0$ if and only if (2.3.12) is satisfied. A similar assertion holds for u^ε.

Proposition 3.2. *If the source term f^ε has expansion (2.3.2)-(2.3.6) and satisfies (2.3.12), then the profile U_0 satisfies*

$$\big[\overline{U}_0\big]_{x_d=0} \;+\; \big[\widetilde{U}_0\big]_{z=0} \;=\; 0\,.$$

In this case, the piecewise smooth leading term $U_0(t,x,x_d/\varepsilon)$ is continous across $\{x_d = 0\}$.

Proof. The ingredients in the analysis are transport equations for the jumps in $\overline{U}_0$ and $\widetilde{U}_0$ separately. To derive the first start with

$$L(\overline{U}_0^\pm) + G(\overline{U}_0^\pm) = \overline{F}_0^\pm\,, \qquad \big[(I-\underline{\pi})\overline{U}_0\big] \;=\; 0\,.$$

Thus at $x_d = 0$ the last $N-k$ components as well as their tangential derivatives are continuous. Multiplying the differential equation for $\overline{U}_0^\pm$ on the left by $\underline{\pi}$ and subtracting values at $x_d = 0\pm$ yields

$$\underline{\pi}\, L\underline{\pi}\,\big[\overline{U}_0^\pm\big] + \underline{\pi}\big[G(\overline{U}_0^\pm)\big] \;=\; \underline{\pi}\big[\overline{F}_0^\pm\big]\,. \tag{3.17}$$

At the same time evaluate (3.15) at $z = 0$ and take differences as $z = 0\pm$ to find

$$\underline{\pi}L(t,x',0,\partial_{t,x})\underline{\pi}\,\big[\widetilde{U}_0^\pm\big] + \underline{\pi}\Big(\big[G(\overline{U}_0^\pm|_{x_d=0\pm} + \widetilde{U}_0^\pm|_{z=0\pm})\big] - \big[G(\overline{U}_0^\pm|_{z=0\pm})\big]\Big) \;=\; \underline{\pi}\big[\widetilde{F}_0^\pm\big]\,, \tag{3.18}$$

Denote by

$$X \;:=\; \underline{\pi}\,L\,\underline{\pi} \;=\; \underline{\pi}\big(\partial_t + \mathbf{v}(t,x').\partial_{x'}\big) + \text{order zero terms}\,.$$

Let

$$w(t,x') \;:=\; \big[\overline{U}_0\big]_{x_d=0} \;+\; \big[\widetilde{U}_0\big]_{z=0} \;=\; \underline{\pi}\,w\,.$$

Adding the transport equations for $\underline{\pi}[\overline{U}_0]_{x_d=0} = [\overline{U}_0]_{x_d=0}$ and $\underline{\pi}[\widetilde{U}_0]_{z=0} = [\widetilde{U}_0]_{z=0}$ the jump in $G(\overline{U}_0)$ terms cancel yielding

$$X\,w + \underline{\pi}\big[G(\overline{U}_0^\pm|_{x_d=0\pm} + \widetilde{U}_0^\pm|_{z=0\pm})\big] \;=\; \underline{\pi}\Big([\overline{F}_0]_{x_d=0} + [\widetilde{F}_0]_{z=0}\Big) \;=\; 0\,.$$

The last equality uses (2.3.12).

Using the definition of G_1 and the fact that $w = \underline{\pi}w$ transforms the nonlinear term to yield

$$X\,w + \underline{\pi}G_1(\overline{U}_0^-|_{x_d=0\pm} + \widetilde{U}_0^-|_{z=0\pm}, w)\,w \;=\; 0\,, \qquad w = 0 \quad \text{when} \quad t < 0\,.$$

The G_1 coefficient is unknown but smooth so this is a linear homogeneous transport equation for w with vanishing initial data. It follows that $w = 0$. ∎

At this point the function $U_0(t,x,z)$ is determined, and with that determination one has

$$W_{-1}^\pm \;=\; \overline{W}_0^\pm \;=\; \underline{\pi}\widetilde{W}_0^\pm \;=\; 0\,.$$

The crunch described in §2.2 occurs when one tries to determine U_1 so that $(I-\underline{\pi})\widetilde{W}_0^\pm = 0$. In $\pm x_d \geq 0$, we have from (3.7) and (3.14),

$$(I-\underline{\pi})\widetilde{W}_0^\pm \;=\; A_d(t,x',0)\partial_z \widetilde{U}_1^\pm + (I-\underline{\pi})\Big(L(t,x',0,\partial)\widetilde{U}_0^\pm + G_1(\overline{U}_0^\pm|_{x_d=0}, \widetilde{U}_0^\pm)\widetilde{U}_0^\pm - \widetilde{F}_0^\pm\Big)\,.$$

Setting this equal to zero and integrating from $z=\pm\infty$ yields for $\pm z \geq 0$

$$(I-\underline{\pi})\widetilde{U}_1^+ = \int_z^\infty Q\Big((I-\underline{\pi})\big(L\,\widetilde{U}_0^+ + G_1(\overline{U}_0^\pm|_{x_d=0}\widetilde{U}_0^\pm), \widetilde{U}_0^\pm - \widetilde{F}_0^+\Big)\,dz\,, \tag{3.19}$$

$$(I-\underline{\pi})\widetilde{U}_1^- = -\int_{-\infty}^z Q\Big((I-\underline{\pi})\big(L\,\widetilde{U}_0^- + G_1(\overline{U}_0^\pm|_{x_d=0}, \widetilde{U}_0^\pm)\widetilde{U}_0^\pm - \widetilde{F}_0^-\Big)\,dz\,. \tag{3.20}$$

No moment condition is needed.

This sets the stage for a recurrence. The equations $W_j^\pm = 0$ for $j=0,1,\ldots,k$ are satisfied by imposing profile equations for $U_0,\ldots,U_{k-1},\underline{\pi}\widetilde{U}_k$.

To see the pattern we continue to complete the determination of U_1. The equation $\overline{W}_1 = 0$ from (3.11) yields

$$L(\partial_{t,x})\overline{U}_1^\pm + G'(\overline{U}_0^\pm)\overline{U}_1^\pm + H(\overline{U}_0^\pm) \;=\; \overline{F}_1^\pm \qquad \text{when} \quad \pm\, x_d \geq 0\,. \tag{3.21}$$

The equation $\widetilde{W}_1 = 0$ from (3.11) is then

$$L(t,x',0,\partial_{t,x})\widetilde{U}_1^\pm(t,x,z) + A_d(t,x',0)\,\partial_z\widetilde{U}_2^\pm + \underline{\pi}G'(U_0^\pm|_{x_d=0})\widetilde{U}_1^\pm + z\partial_d A_d(t,x',0)\partial_z\widetilde{U}_1^\pm = \widetilde{F}_j^\pm\,.$$

Place the already determined $(I-\underline{\pi})\widetilde{U}_1^\pm$ with the source terms and multiply by $\underline{\pi}$ to find with $X := \underline{\pi}L\underline{\pi}$

$$\big(X + z\partial_d A_d(t,x',0)\partial_z\big)\underline{\pi}\widetilde{U}_1 + \underline{\pi}G'(U_0^\pm|_{x_d=0})\widetilde{U}_1^\pm \;=\; \underline{\pi}\widetilde{F}_1^\pm - \underline{\pi}\,L\,(I-\underline{\pi})\widetilde{U}_1^\pm\,.$$

This is an inhomogeneous linear tranport equation for $\underline{\pi}\widetilde{U}_1^\pm$ parallel to $z=0$. Adjoining the initial condition

$$\underline{\pi}\,\widetilde{U}_1^\pm\big|_{t<0} \;=\; 0\,,$$

the function $\underline{\pi}\widetilde{U}_1^\pm$ is uniquely determined.

Aside. Typically U_j is discontinuous across $x_d=0$ and $\widetilde{U}_j$ is discontinuous across $z=0$. However when f^ε is smooth, we show in Corollary 6.2 that modulo an infinitely small modification, u^ε is smooth.

With $\widetilde{U}_1^\pm$ in hand, the $j=1$ case of the transmission condition (3.1) yields

$$\big[(I-\underline{\pi})\overline{U}_1^\pm\,\big]_{x_d=0} \;=\; -\big[\widetilde{U}_1\big]_{z=0}\,. \tag{3.22}$$

From (3.19)-(3.20), the right hand side is known. Therefore $\overline{U}_1$ is determined as the unique smooth solution of the inhomogeneous transmission problem (3.21-22) which vanishes for $t<0$. We recall that to solve this transmission problem one observes that the system (3.21-22) is equivalent to the system

$$L\overline{U}_1 + G'(\overline{U}_0)\overline{U}_1 + H(\overline{U}_0) = -A_d[\widetilde{U}_1]_{\{z=0\}}\,\delta(x_d) + \overline{F}_1.$$

Assume that the right hand side is given on a domain $]-\infty,T]\times\mathbb{R}^d$. Choose $v\in L^2(]-\infty,T]\times\mathbb{R}^d)$ such that $v_\pm := v|_{\pm x_d>0} \in H^\infty(]-\infty,T]\times\mathbb{R}^d_\pm)$ and $[v]_{x_d=0} = [\widetilde{U}_1]_{z=0} \in \ker \underline{\pi}$, with $v|_{t<0}=0$. Construct the solution $\overline{U}_1$ as $\overline{U}_1 = v+w$ where w satisfies

$$Lw + G'(\overline{U}_0)w + H(\overline{U}_0) = \overline{F}_1 + g \quad \text{on }]-\infty,T]\times\mathbb{R}^d, \qquad w|_{t<0}=0\,.$$

Here g is a piecewise smooth function equal to $Lv_\pm + G'(\overline{U}_0)v_\pm + H(\overline{U}_0)_\pm$ on $\pm x_d>0$. This system for w is a *continuation problem* for a linear hyperbolic system with a discontinuous piecewise-H^∞ source term, which admits a unique and piecewise-H^∞ solution following classical results on the propagation of singularities in hyperbolic systems ([B], [RR1,2], [M1,2]). We use in a strong way the transmission structure of the conditions (3.22).

The problem is **not** treated as a characteristic boundary value problem. It is treated as an inhomogeneous initial value problem. This is an important point because at present there is no general theory of the characteristic boundary value problem for strictly hyperbolic systems. However, for **symmetric systems** such results are available ([R], [MO], [G1]2]), and this boundary value problem approach, followed in §7, yields complementary results.

Having now determined U_0, U_1, the equations $W_{-1}=W_0=\overline{W}_1=\underline{\pi}\widetilde{W}_1=0$ are satisfied as well as the cases $j=0,1$ of the transmission condition (3.9). The inductive definition of the profiles continues by setting $(I-\underline{\pi})\widetilde{W}_1=0$ determining $(I-\underline{\pi})\widetilde{U}_2$, and so forth.

The data for the next proposition is the following.

1. A sequence of functions $\overline{F}_j^\pm(t,x)$ indexed by $j\geq 0$ and $\pm$, smooth on $]-\infty,T_0]\times\{x_d\geq 0\}$, supported in $\{t\geq 0\}$ and compactly supported in x.

2. A sequence of function $\widetilde{F}_j^\pm(t,x',z)$ indexed by $j\geq 0$ and $\pm$, smooth on $]-\infty,T_0]\times\mathbb{R}^{d-1}\times\{\pm z\geq 0\}$, supported in $\{t\geq 0\}$, compactly supported in x' and rapidly decreasing as $\pm z\to\infty$.

Denote by

$$\mathcal{O}_T^+ := \{(t,x)\in\mathbb{R}\times\mathbb{R}^d \,:\, t<T,\ x_d>0\} \quad\text{and}\quad \mathcal{O}_T^- := \{(t,x)\in\mathbb{R}\times\mathbb{R}^d \,:\, t<T,\ x_d<0\}\,.$$

Proposition 3.3. Existence and unicity of profiles. *There is a $0<T_1\leq T_0$ and a unique maximal solution $\overline{U}_0\in L^\infty_{\rm loc}\big(]-\infty,T_1[\times\mathbb{R}^d\big)$ to*

$$L(\overline{U}_0) \;+\; G(\overline{U}_0) \;=\; \overline{F}\,, \qquad \overline{U}_0 \;=\; 0 \quad\text{when}\quad t<0\,.$$

The solution is piecewise smooth and for any $T<T_1$ the restriction $\overline{U}_0^\pm$ to $\mathcal{O}_T^\pm$ is in $H^\infty(\mathcal{O}_T^\pm)$ and compactly supported.

There is a $T_2\in]0,\ T_1]$ and a unique maximal solution $\widetilde{U}_0^\pm\in C^\infty_{\rm loc}\big(]-\infty,T_2[\times\mathbb{R}^{d-1}\times\{\pm z\geq 0\}\big)$ to (3.17-18). The layer profiles $\widetilde{U}_0^+$ and $\widetilde{U}_0^-$ are compactly supported in x' and rapidly decreasing as $\pm z\to\infty$ uniformly on compact time intervals.

If $T_3\in]0,T_2[$ and $j\geq 1$, then there are uniquely determined $\overline{U}_j^\pm(t,x)\in H^\infty\big(\mathcal{O}_{T_3}^\pm\big)$ and $\widetilde{U}_j^\pm(t,x',z)\in H^\infty\big(]-\infty,T_3]\times\{\pm z\geq 0\}\big)$ which satisfy the profile equations derived above. They are compactly supported in x and the $\widetilde{U}_j^\pm$ are rapidly decreasing as $\pm z\to\infty$.

One of the most striking aspects of this construction is that the transport operator $\mathbb{H}$ has a different vector field than the standard transport in geometric optics. One could think that the new $z\partial_z$

term represents a fundamentallly new effect. The next result shows that in fact it can be eliminated by choosing local coordinates so that the hyperplanes $\{x_d = \text{const.}\}$ are all characteristic. This can be done, for example by defining $\psi(t,x)$ locally as the solution of the eikonal equation

$$\partial_t \psi \;=\; \tau(t,x,\nabla_x\psi)\,, \qquad \psi(0,x) = x_d\,.$$

In the new coordinates $\tilde{x}(t,x) := (x', \psi(t,x))$, $\tilde{t} = t$, the hyperplanes $\tilde{x}_d = \text{const.}$ are characteristic.

Proposition 3.4. *If the hyperplanes* $x_d = \text{const.}$ *are characteristic, then the coefficient* $\underline{\pi}\partial_d A_d \underline{\pi}$ *of the* $z\partial_z$ *term in* $\mathbb{H}$ *vanishes identically.*

Proof. The algebraic lemma of geometric optics implies that

$$\underline{\pi}\;\partial_d A_d\;\underline{\pi} \;=\; \frac{\partial \tau(t,x',0\,;0,\ldots,0,1)}{\partial x_d}\,.$$

Since the hyperlanes are characteristic it follows that $\tau(t,x',x_d;0,\ldots,0,1) = 0$. Differentiating with respect to x_d proves the Proposition. ∎

§4. Approximate solution and residual estimates.

Suppose that T_3, F_j, and U_j are as in the above proposition. Borel's theorem provides functions $F^\varepsilon(t,x,z)$ and $U^\varepsilon(t,x,z)$ compactly supported in x, so that in $[0,T_3]\times\{\pm x_d \ge 0\}\times\{\pm z\ge 0\}$

$$F^\varepsilon \;\sim\; \sum_{j=0}^{\infty} \varepsilon^j\, F_j^\pm(t,x,z)\,, \qquad U^\varepsilon \;\sim\; \sum_{j=0}^{\infty} \varepsilon^j\, U_j^\pm(t,x,z)\,. \tag{4.1}$$

Define sources and approximate solutions by

$$f^\varepsilon(t,x) \;:=\; F^\varepsilon(t,x,x_d/\varepsilon)\,, \qquad u^\varepsilon(t,x) \;:=\; U^\varepsilon(t,x,x_d/\varepsilon)\,. \tag{4.2}$$

Thanks to (3.1), the function U^ε can be chosen so that $(I-\underline{\pi})u^\varepsilon$ is continuous across $\{x_d = 0\}$.

Aside. The source term can be chosen continuous if and only if the profiles F_j satisfy for all j,

$$\overline{F}_j^{+}(t,x',x_d=0) + \widetilde{F}_j^{+}(t,x',z=0) \;=\; \overline{F}_j^{-}(t,x',x_d=0) + \widetilde{F}_j^{-}(t,x',z=0)\,. \tag{4.3}$$

There are analogous necessary and sufficient conditions for membership in C^k.

Denote by $u^\varepsilon_{\text{exact}}$ the solution of

$$L\,u^\varepsilon_{\text{exact}} + G(u^\varepsilon_{\text{exact}}) \;=\; f^\varepsilon\,, \qquad u^\varepsilon_{\text{exact}} = 0 \quad \text{when} \quad t<0\,. \tag{4.4}$$

Define the error $E^\varepsilon(t,x)$ by

$$E^\varepsilon(t,x) \;:=\; u^\varepsilon_{\text{exact}} - u^\varepsilon \tag{4.5}$$

Similarly define the residual $r^\varepsilon(t,x)$ by

$$r^\varepsilon(t,x) \;:=\; Lu^\varepsilon + G(u^\varepsilon) - f^\varepsilon\,. \tag{4.6}$$

The next Proposition is an immediate consequence of the construction of the preceding section.

Proposition 4.1 *When the profiles U_j are constructed satsifying the equations of §3, the residual r^ε satisfies the conormal estimates*

$$\forall N \geq 0, \;\; \forall \alpha, \;\; \forall p \in [2,\infty], \quad \big\|(\partial_{t,x'}\,,\, \mathcal{Z})^\alpha r^\varepsilon\big\|_{L^p([0,T_3]\times\mathbb{R}^d)} \;=\; O(\varepsilon^N)\,, \tag{4.7}$$

and piecewise estimates

$$\forall N \geq 0, \;\; \forall \beta, \;\; \forall p \in [2,\infty], \quad \big\|(\partial_{t,x'}\,,\, \mathcal{Z},\, \varepsilon\partial_d)^\beta r^\varepsilon\big\|_{L^p([0,T_3]\times\{\pm x_d>0\})} \;=\; O(\varepsilon^N)\,. \tag{4.8}$$

When the sources f^ε are smooth, $u^\varepsilon_{\rm exact}$ is smooth. In this case, the approximate solution is piecewise smooth and Lemma 3.1 shows that the jump in the approximate solution vanishes to leading order. In Corollary 6.2 we show that u^ε is smooth to infinite order in the sense that the jumps in the derivatives of u^ε have asymptotic expansions all of whose terms vanish, so that there is an equivalent family $\underline{u}^\varepsilon$ which is smooth.

§5. Conormal existence and stability.

The next two results are consequences of standard conormal theory ([RR1], [RR2], [M1], [M2]) applied to the transmission problem (2.3.1). If Ω is an open set in $\mathbb{R}\times\mathbb{R}^d$, denote by

$$H^m_{co}(\Omega) \;:=\; \Big\{ u \in L^2(\Omega) \;:\; \forall |\alpha| \leq m, \;\; (\partial_t, \partial_{x'}, \mathcal{Z})^\alpha u \in L^2(\Omega) \Big\}\,,$$

equipped with the natural norm

$$\|u\|^2_{H^m_{co}(\Omega)} \;:=\; \sum_{|\alpha|\leq m} \|(\partial_t, \partial_{x'}, \mathcal{Z})^\alpha u\|^2_{L^2(\Omega)}\,.$$

If $\mathrm{dist}(\Omega, \{x_d = 0\}) > 0$ then $H^m_{co}(\Omega)$ coincides with the usual Sobolev space $H^m(\Omega)$. When $\overline{\Omega}$ intersects $\{x_d = 0\}$, the functions in $H^m_{co}(\Omega)$ are allowed to be singular on $\{x_d = 0\}$. If $U(t,x,z)$ is a profile as in Section 3, defined on $\Omega =]-\infty, T]\times\mathbb{R}^d\times\mathbb{R}_z$, the function $v^\varepsilon(t,x) = U(t,x,x_d/\varepsilon)$ belongs to $H^m_{co}(\Omega)$ for every $\varepsilon > 0$ fixed (recall that in general $U(t,x,z)$ is not continuous across $\{x_d = 0\}$). Moreover, v^ε is uniformly bounded in $H^m_{co}(\Omega)$ as $\varepsilon \to 0$,

$$\sup_{0<\varepsilon\leq 1} \big\|U(t,x,x_d/\varepsilon)\big\|_{H^m_{co}(\Omega)} \;<\; \infty.$$

An important result concerning conormal waves in semilinear hyperbolic systems is that the continuation problem is well posed in the space of *bounded and conormal fonctions*, $L^\infty \cap H^m_{co}$. Another important point is that once a solution is in H^{2m}_{co}, then regularity for m normal derivatives propagate on each half space $\{\pm x_d \geq 0\}$, following the rule[1] "one normal derivative costs two conormal derivatives". We can take $m = \infty$, so the solutions are piecewise H^∞. This is rigourously stated in the next proposition and proved in [RR1],[M1].

Proposition 5.1 *Suppose that $0 < T < \infty$ and the source $f(t,x)$ is piecewise smooth and compactly supported in x. Then the transmission problem (2.3.1) has a unique maximal solution*

[1] More precisely, the propagation of regularity holds in the space defined by $\partial_d^k u \in H^{2m-2k}_{co}$, $k = 0,\cdots,m$.

$u\in L^\infty_{\rm loc}(]-\infty,T^*[\times\mathbb{R}^d)$ *which for any* $T<T^*$ *is piecewise smooth and compactly supported in* $]-\infty,T]\times\mathbb{R}^d$.

In order to apply the next result to the transmission problem (3.3.1) as well as to the error equation, consider a slightly more general equation of the form

$$Lu+F(a,u) \ = \ f \quad \text{in }]-\infty,T]\times\mathbb{R}^d, \qquad u\big|_{t<0} \ = \ 0,$$

where $F(.\,,.)$ is a smooth function of $(a,u)\in\mathbb{R}^{N'}\times\mathbb{R}^N$. The function a is a parameter of the problem, and, $F(\cdot,0)=0$. In the next proposition let

$$\mathcal{O}_T \ := \ ]-\infty,T]\times\mathbb{R}^d\,.$$

Theorem 5.2 ([RR1], [M2]) *There is an* $m(d)<\infty$ *so that if* $\mathbb{N}\ni m\geq m(d)$, $T>0$, *and* $a,f\in L^\infty(\mathcal{O}_T)\cap H^m_{co}(\mathcal{O}_T)$, *with* $f|_{t<0}=0$, *then there is* $T'>0$ *and a unique* $u\in L^\infty(\mathcal{O}_{T'})\cap H^m_{co}(\mathcal{O}_{T'})$ *satisfying*

$$Lu+F(a,u)=f \quad \text{in } \mathcal{O}_{T'}, \quad \text{and } u\big|_{t<0}=0.$$

In addition there is a there is a constant $c=c(R)$ *so that if* $\|a\|_{L^\infty(\mathcal{O}_T)}+\|a\|_{H^m_{co}(\mathcal{O}_T)}\leq R$, *then*

$$\|u\|_{L^\infty(\mathcal{O}_{T'})}+\|u\|_{H^m_{co}(\mathcal{O}_{T'})} \ \leq \ c_m(R)\left(\|f\|_{L^\infty(\mathcal{O}_{T'})}+\|f\|_{H^m_{co}(\mathcal{O}_{T'})}\right). \tag{5.1}$$

Moreover, if the quantity $\|f\|_{L^\infty(\mathcal{O}_T)}+\|f\|_{H^m_{co}(\mathcal{O}_T)}\leq R$ *is small enough one can take* $T'=T$. *Finally, if* a *and* f *belong to* $H^{m'}_{co}(\mathcal{O}_T)$ *with* $m'>m$, *then the life span of the* $L^\infty\cap H^m_{co}$ *solution and of the* $L^\infty\cap H^{m'}_{co}$ *solution are the same.*

Proposition 5.1 follows from Theorem 5.2 applied with $m\to\infty$.

§6. Validity of the asymptotic expansion.

By construction, the error E^ε defined in (4.5) satisfies

$$LE^\varepsilon+G_1(u^\varepsilon,E^\varepsilon)E^\varepsilon=-r^\varepsilon\,, \qquad \text{and} \qquad E^\varepsilon\big|_{t<0}=0. \tag{6.1}$$

Theorem 5.2 applied to system (6.1) together with Proposition 4.1 imply the following theorem, which is our main result.

Theorem 6.1 *Suppose that* T_3, *profiles* U_j, F_j, *source* f^ε *and approximate solution* u^ε *are as in §3 and Proposition 4.1. Then there is an* $\varepsilon_0>0$ *so that for* $0<\varepsilon<\varepsilon_0$ *the transmission problem with source* f^ε *has a unique compactly supported piecewise smooth solution* $u^\varepsilon_{\rm exact}$ *on* $]-\infty,T_3]\times\mathbb{R}^d$. *The error is infinitely small in the sense that*

$$\forall N>0,\ \ \forall\alpha,\ \ \forall p\in[2,\infty], \quad \big\|(\partial_{t,x'}\,,\,\mathcal{Z})^\alpha\big(u^\varepsilon-u^\varepsilon_{\rm exact}\big)\big\|_{L^p([0,T_3]\times\mathbb{R}^d)} \ = \ O(\varepsilon^N)\,, \tag{6.2}$$

and

$$\forall N>0,\ \ \forall\beta,\ \ \forall p\in[2,\infty], \quad \big\|\partial^\beta\big(u^\varepsilon-u^\varepsilon_{\rm exact}\big)\big\|_{L^p([0,T_3]\times\{\pm x_d>0\})} \ = \ O(\varepsilon^N)\,. \tag{6.3}$$

Proof. a. The existence of E^ε on $]-\infty, T_3]$ follows from Theorem 5.2 and from the fact that r^ε is $O(\varepsilon^\infty)$ in L^∞ and in H^m_{co}. The estimates (6.2) with $p = 2$ are direct consequences of the estimates (5.1) in Theorem 5.2.

b. For $p = 2$ again, the estimates (6.3) follow from the classical principle of "one normal derivative for two conormal derivatives", that we do not detail here. Just note that each normal derivative of u^ε causes a loss of one power of ε in the usual estimate, hence a loss of ε^k in the estimation of $\partial_d^k E^\varepsilon$, leading to the following estimate

$$\|(\partial_{t,x'}, \, , \mathcal{Z} \, , \varepsilon\partial_d)^\alpha E^\varepsilon\|_{L^2(\mathcal{O}_T\cap\{\pm x_d>0\})} \le O(\varepsilon^N) \, .$$

Since N is arbitrary, the estimate (6.3) follows.

c. The estimates with $p = \infty$ are obtained by using the estimates with $p = 2$ and Sobolev's inequality. ∎

Remark. This shows that the L^∞ error in the leading term approximation is $O(\varepsilon)$. That this cannot be improved follows from the fact that the correctors are $O(\varepsilon)$.

Proposition 3.2 proved that if f^ε is continuous to leading order, then the same is true of u^ε. The next result generalizes this to infinite order. The interested reader can fill in the finite order results and thereby give an independent proof of Proposition 3.2.

Corollary 6.2. *Suppose that f^ε is smooth across across $\{x_d = 0\}$. Then*

$$\forall k \, , \quad \forall N > 0 \, , \quad \left[\frac{\partial^k u^\varepsilon}{\partial x_d^k}\right]_{x_d=0} = O(\varepsilon^N) \, .$$

There is a possibly different choice of $\underline{U}^\varepsilon$ with the same asymptotic expansion so that the approximate solution $\underline{u}^\varepsilon := \underline{U}^\varepsilon(t, x, x_d/\varepsilon)$ is smooth.

Proof. Differentiating the expansions for u^ε in $\pm x_d > 0$ shows that the jump has an asymptotic expansion

$$\Big[(\varepsilon\partial_{x_d})^k u^\varepsilon\Big]_{x_d=0} \sim \sum_{j=0}^{\infty} \varepsilon^j \, V_j(t, x')$$

with V_j determined by the germ of $\overline{U}_j^\pm$ at $x_d = 0$ and the germ of $\widetilde{U}_j^\pm$ at $z = 0$.

The smoothness of f^ε implies the smoothness of $u^\varepsilon_{\rm exact}$ so that for all k the jump in $\partial^k_{x_d} u^\varepsilon_{\rm exact}$ vanishes. Theorem 6.1 then implies that the jump in $\partial^k_{x_d} u^\varepsilon$ is $O(\varepsilon^\infty)$.

It follows that all of the V_j vanish identically. The standard constructive Borel summation argument yields $\underline{U}^\varepsilon$ and therefore $\underline{u}^\varepsilon$ for which these jumps vanish (see for example [RK]). ∎

Examples. The sources at the start of §2, and also the mollified sources from Proposition 2.2 satisfy the hypotheses of this Corollary.

§7. The initial boundary value problem approach.

In this section we assume that the system L is *symmetric hyperbolic*. Instead of introducing transmission problems we treat an equivalent boundary value problem in the domain $\{x_d > 0\}$. On $\{x_d > 0\}$ let

$$v^\varepsilon(t, x', x_d) \; := \; \Big(u^\varepsilon(t, x', x_d) \, , \, u^\varepsilon(t, x', -x_d)\Big), \quad x_d > 0 \, .$$

For the unknown v^ε, we have the hyperbolic mixed initial boundary value problem

$$L^\sharp v^\varepsilon + G^\sharp(v^\varepsilon) = f^\sharp \quad \text{in} \quad \{t < T\} \times \mathbb{R}^{d-1} \times \{x_d > 0\}\,, \qquad v^\varepsilon\big|_{t<0} = 0\,, \tag{7.1}$$

with boundary conditions

$$M v^\varepsilon = 0 \quad \text{on} \quad \{x_d = 0\}, \qquad M := \begin{pmatrix} I - \underline{\pi} & -I + \underline{\pi} \\ 0 & 0 \end{pmatrix}. \tag{7.2}$$

The operator $L^\sharp$ is $\partial_t + \sum A_j^\sharp \partial_j + B^\sharp$, with

$$A_j^\sharp(t,x) := \begin{pmatrix} A_j & 0 \\ 0 & A_j(t,x',-x_d) \end{pmatrix}, \qquad A_d^\sharp := \begin{pmatrix} A_d & 0 \\ 0 & -A_d(t,x',-x_d) \end{pmatrix}. \tag{7.3}$$

The definitions of $G^\sharp$, $B^\sharp$ and $f^\sharp$ are clear.

The large system is symmetric, and the boundary conditions (7.2) are *maximally dissipative* in the sense of Lax and Phillips [L.P], (see also [A], [S]). Formula (1.2) shows that the boundary $\{x_d = 0\}$ is characteristic of constant multiplicity for the operator $L^\sharp$. The results of Rauch and Guès ([R], [G1], [G2]) apply to the system (7.1)-(7.2) which therefore has a unique solution in $H^\infty(]-\infty, T] \times \mathbb{R}^d_+)$, for some $T > 0$, independent of $\varepsilon \in]0,1]$.

More generally, it is natural to consider the hyperbolic initial boundary value problem with general maximal dissipative boundary conditions. For $T > 0$, denote

$$\Omega_T \ := \ \big\{(t,x) \in \mathbb{R}^{1+d} \, : \, t < T, \ x_d > 0\big\}\,, \qquad \text{and} \qquad \Gamma_T \ := \ \big\{(t,x) \in \mathbb{R}^{1+d} \, : \, t < T, \ x_d = 0\big\}\,.$$

Returning to the notations of §1, consider the problem

$$\left\{ \begin{array}{rcl} Lu^\varepsilon + G(u^\varepsilon) &=& f^\varepsilon \quad \text{in} \quad \Omega_T, \\ u^\varepsilon &\in& \mathcal{N} \quad \text{on} \quad \Gamma_T, \\ u^\varepsilon\big|_{t<0} &=& 0. \end{array} \right. \tag{7.4}$$

where $\mathcal{N}(t,x')$ is a smoothly varying maximal negative subspace of $\mathbb{R}^N$ for the quadratic form $\langle A_d(t,x',0)\cdot\,,\,\cdot\rangle$. In contrast with §1, f^ε is defined only in $\{x_d > 0\}$.

Define $\mathbf{P}(\Omega_T)$ to be the set of fonctions $\mathbf{V} : \Omega_T \to \mathbb{R}$ of the form

$$\mathbf{V}(t,x) \ = \ \overline{\mathbf{V}}(t,x) + \widetilde{\mathbf{V}}(t,x',z)$$

with $\overline{\mathbf{V}} \in H^\infty(\Omega_T)$ and $\widetilde{\mathbf{V}} \in H^\infty(\Gamma_T \times [0,+\infty[)$ with *rapid decay* to 0 when $z \to 0$ for the tilde part,

$$\forall p \in \mathbb{N}, \forall k \in \mathbb{N}, \forall \alpha \in \mathbb{R}^d, \forall \ell \in \mathbb{N} : \quad \sup_{\Omega_T} \left| z^p \, \partial_t^k \partial_{x'}^\alpha \partial_z^\ell \widetilde{\mathbf{V}} \right| \ < \ \infty.$$

We assume that

$$f^\varepsilon \ = \ \mathbf{F}^\varepsilon(t,x,x_d/\varepsilon)$$

with

$$\mathbf{F}^\varepsilon(t,x,z) \ \sim \ \sum_{j=0}^{\infty} \varepsilon^j \, \mathbf{f}_j(t,x,z), \quad \text{with} \quad \mathbf{f}_j \in \mathbf{P}(\Omega_{T_0})$$

for some given $T_0 > 0$, and with

$$\mathbf{f}_j\big|_{t<0} \;=\; 0\,.$$

We seek a response u^ε defined on Ω_T for some $T > 0$ independant of ε, and of the form

$$u^\varepsilon(t,x) \;\sim\; \sum_{j=0}^{\infty} \varepsilon^j \mathbf{V}_j(t,x,x_d/\varepsilon), \quad \text{where} \quad \mathbf{V}_j \in \mathbf{P}(\Omega_T).$$

Introduce the solution of the $\varepsilon = 0$ limit problem. By the results of [G1], [S], there exists $T_1 > 0$ and a unique $\overline{\mathbf{V}^0} \in H^\infty(\Omega_{T_1})$ satisfying

$$\begin{cases} L\overline{\mathbf{V}^0} + G(\,\overline{\mathbf{V}^0}\,) = \overline{\mathbf{f}^0} \quad \text{in} \quad \Omega_{T_1}, \\ \overline{\mathbf{V}^0} \;\in \mathcal{N} \quad \text{on} \quad \Gamma_{T_1}, \\ \overline{\mathbf{V}^0}\big|_{t<0} \;=\; 0. \end{cases} \tag{7.5}$$

The operator $\mathbb{H}$ is as in §1.

Define (for some $0 < T_2 < T_1$), $\widetilde{\mathbf{V}^0} \in H^\infty(\Gamma_{T_2} \times \mathbb{R}_+)$ as the unique solution on $\Gamma_{T_2} \times \mathbb{R}_+$ of the system

$$\begin{cases} (I - \underline{\pi})\widetilde{\mathbf{V}^0} \;=\; 0, \\ \mathbb{H}\,\widetilde{\mathbf{V}^0} + \underline{\pi}\Big(G(\,\overline{\mathbf{V}^0} + \widetilde{\mathbf{V}^0}\,) - G(\,\overline{\mathbf{V}^0}\,)\Big) \;=\; \underline{\pi}\widetilde{\mathbf{f}^0} \quad \text{in} \quad \Gamma_{T_2} \times \mathbb{R}_+, \;. \\ \widetilde{\mathbf{V}^0}\big|_{t<0} \;=\; 0. \end{cases} \tag{7.6}$$

There is no need of boundary conditions on $\{z = 0\}$ since the boundary $\{z = 0\}$ is totally characteristic for the operator $\mathbb{H}$.

Theorem 7.1. *Let $\mathbf{V}^0 := \overline{\mathbf{V}^0} + \widetilde{\mathbf{V}^0}$, with $\overline{\mathbf{V}^0}$ and $\widetilde{\mathbf{V}^0}$ defined above. For ε small enough, the system (7.4) has a unique solution $u^\varepsilon \in H^\infty(\Omega_{T_2})$ on Ω_{T_2} and*

$$\forall\alpha, \qquad \Big\|(\partial_{t,x'}\,,\,\mathcal{Z}\,,\,\varepsilon\partial_d)^\alpha(u^\varepsilon - \mathbf{V}^0(t,x,x_d/\varepsilon)\Big\|_{L^2(\Omega_{T_2})\cap L^\infty(\Omega_{T_2})} \;=\; O(\varepsilon)\,.$$

The structure of the proof is the same as for the Main Theorem in §1. The first step is to construct an approximate solution

$$u_a^\varepsilon(t,x) \;\sim\; \sum_{0}^{\infty} \varepsilon^j\,\mathbf{V}_j(t,x,x_d/\varepsilon)$$

on Ω_{T_2}. The first profile $\mathbf{V}^0$ is found by solving the profile equations (7.5) (which gives $\overline{\mathbf{V}}^0$ and T_1) and (7.6) (which gives $\widetilde{\mathbf{V}}^0$ and T_2). The other profiles $\mathbf{V}_j$, $j \geq 1$, are then obtained by induction, solving *linear* well posed problems on the domain $\Omega_{T_2} \times \mathbb{R}_+$. The resulting approximate solution satisfies

$$\begin{cases} Lu_a^\varepsilon + G(u_a^\varepsilon) \;=\; f^\varepsilon + r^\varepsilon \quad \text{in} \quad \Omega_{T_2}\,, \\ u_a^\varepsilon \;\in \mathcal{N} \quad \text{on} \quad \Gamma_T\,, \\ u_a^\varepsilon\big|_{t<0} \;=\; 0\,. \end{cases}$$

where $\|\partial^\alpha r^\varepsilon\|_{L^2(\Omega_{T_2})} = O(\varepsilon^k)$, for all $\alpha \in \mathbb{N}^{d+1}$, and $k \in \mathbb{N}$.

The second step is to look for the exact solution u^ε as $u^\varepsilon = u_a^\varepsilon + w^\varepsilon$ where w^ε is the solution of

$$\begin{cases} Lw^\varepsilon + G_1(u_a^\varepsilon, w^\varepsilon)w^\varepsilon \;=\; -r^\varepsilon \quad \text{in} \quad \Omega_{T_2}\,, \\ w^\varepsilon \;\in \mathcal{N} \quad \text{on} \quad \Gamma_T\,, \\ w^\varepsilon\big|_{t<0} \;=\; 0\,, \end{cases} \tag{7.7}$$

where $G_1(a,b)b := G(a+b) - G(a)$ as in formula (3.6). In this system, the functions a^ε are bounded in the space $L^\infty(\Omega_{T_2}) \cap H^m_{co}(\Omega_{T_2})$, whose definition was recalled in Section 5. The results of [G1],[G2] apply to the problem (7.7) showing that for $0 < \varepsilon < \varepsilon_0$ with ε_0 small enough, there exits a unique solution $w^\varepsilon \in H^\infty(\Omega_{T_2})$, uniformly bounded in $L^\infty(\Omega_{T_2}) \cap H^m_{co}(\Omega_{T_2})$ and which satisfies for all $\alpha \in \mathbb{N}^{d+1}$ and $n \in \mathbb{N}$,

$$\|(\partial_{t,x'}, \mathcal{Z}, \varepsilon\partial_d)^\alpha w^\varepsilon\|_{L^2(\Omega_{T_2}))} \leq c_{\alpha,N}\, \varepsilon^N .$$

Theorem 7.1 follows. ∎

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