Real and Complex Regularity are Equivalent
for Hyperbolic Characteristic Varieties

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If $P(\eta)$ is a real homogeneous polynomial one associates real and a complex algebraic varieties

$$\mathbf{V_R} := \{\eta \in \mathbf{R}^n \setminus 0 \ : \ P(\eta) = 0\} \quad \text{and} \quad \mathbf{V_C} := \{\eta \in \mathbf{C}^n \setminus 0 \ : \ P(\eta) = 0\},$$

with $\mathbf{V_R} \subset \mathbf{V_C}$.

Definition *A homogeneous polynomial is* **hyperbolic** *with timelike direction $\theta \in \mathbf{R}^n \setminus 0$ iff for all real η the equation $P(\eta + s\theta) = 0$ has only real roots s (see [G], [Hö]).*

In the trivial case of P being a constant, both varieties are empty. Taking $\eta = 0$ shows that $P(\theta) \neq 0$.

If P is of degree $m \geq 1$, then for each $\eta \in \mathbf{R}^n$, the equation $P(\eta + s\theta) = 0$ has m real roots counting multiplicity so the line $\eta + s\theta$ cuts the varieties $\mathbf{V}$ in at least 1 and no more than m points. It follows that $\mathbf{V_R}$ (resp. $\mathbf{V_C}$) is a real algebraic variety (resp. algebraic variety) of real (resp. complex) codimension equal to one. $\mathbf{V_R}$ is called the **characteristic variety**.

The fundamental stratification theorems of real and complex algebraic geometry (see [BR], [H])) imply that with the exception of a set of real or complex codimension 2, the varieties $\mathbf{V_R}$ and $\mathbf{V_C}$ are locally real analytic and analytic. That means on a neighborhood of a non exceptional point $\underline{\eta}$ there is a real analytic function (resp. analytic function) $\phi(\eta)$ with $\phi(\underline{\eta}) = 0$ and $d\phi(\underline{\eta}) \neq 0$ whose zero set coincides with the variety.. The non exceptional points are called **regular** according to the next definition.

Definition *For non constant hyperbolic P, a point $\eta \in \mathbf{V_R}$ is a* **regular point** *of $\mathbf{V_R}$ (resp. $\mathbf{V_C}$) iff in an $\mathbf{R}^n$ (resp. $\mathbf{C}^n$) neighborhood of η, $\mathbf{V_R}$ (resp. $\mathbf{V_C}$) is a real analytic (resp. analytic) manifold of real (resp. complex) codimension equal to 1.*

For varieties defined by non hyperbolic polynomials the two notions of regularity are distinct as the following example shows.

Examples. The equation $\eta_1(\eta_1 - i(\eta_2 - 2)^2) = 0$ has a real leaf $\eta_1 = 0$ which touches at $(0, 2)$ the complex leaf $\eta_1 = i(\eta_2 - 2)^2$. The equation

$$\eta_1(\eta_1 - i(\eta_2 - 2)^2)(\eta_1 + i(\eta_2 - 2)^2) = 0$$

is real and has two complex leaves touching the real leaf. The real variety is regular at $(0, 2)$ and the complex variety is not. The equation

$$\eta_1(\eta_1\eta_3 - i(\eta_2 - 2\eta_3)^2)(\eta_1\eta_3 + i(\eta_2 - 2\eta_3)^2) = 0$$

is homogeneous and real. The section $\eta_3 = 1$ gives the previous example. Therefore, $(0, 2, 1)$ is a regular point of the real variety and is not a regular point of the complex variety.

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For hyperbolic polynomials, the two notions are equivalent. This fact is difficult and may be impossible to find in the literature. It can be proved using the Weierstrass Preparation Theorem to examine the zero set as in the analysis of the zeros of an arbitrary real analytic function. In this note we give a short and elementary proof.

The result already has two applications. The equivalence is used by B. Texier [T] in his elegant derivation of the algebraic identities of geometric optics. In this note we show how it yields an algebraic algorithm for computing the germ of $\mathbf{V_R}$ at a regular point η from the germ of P at η. This extends the result of [R] where an independent construction is used to compute the tangent plane of $\mathbf{V_R}$ at η.

Theorem. *If P is a homogeneous hyperbolic polynomial, then $\eta \in \mathbf{V_R}$ is a regular point of $\mathbf{V_R}$ if and only if η is a regular point of $\mathbf{V_C}$.*

Proof. If P is a constant polynomial the result is trivial. It suffices to treat non constant P.

To prove the if assertion, suppose that $\underline{\eta}$ is a regular point of $\mathbf{V_C}$. Express

$$P \ = \ \Pi_\alpha \big(P_\alpha\big)^{m(\alpha)}$$

as a product of irreducible factors. A point $\underline{\eta} \in \mathbf{V_C}$ is regular if and only if it is a root of exactly one of the P_α and for that α, $dP(\underline{\eta}) \neq 0$ (see e.g. [Ha, Thm. I.5.1]).

For $\eta \in \mathbf{R}^n$ and $s \notin \mathbf{R}$ one has $P(\eta + s\theta) \neq 0$ which implies that $P_\alpha(\eta + s\theta) \neq 0$ so each P_α is hyperbolic with θ timelike.

Choose linear coordinates

$$\eta = (\tau, \xi) \ \in \mathbf{R} \times \mathbf{R}^{n-1}\,, \qquad \underline{\eta} = (\underline{\tau}, \underline{\xi})\,,$$

so that $\theta = (1, 0, \ldots, 0)$. Since $P_\alpha(\theta) \neq 0$, multiplying P_α by a constant reduces to the case $P_\alpha = \tau^p + b_1(\xi)\tau^{p-1} + \cdots$. The coefficients of τ^j are elementary symmetric functions of the roots τ of $P_\alpha(\tau, \xi) = 0$. Since those roots are real for real ξ, the $b_j(\xi)$ are polynomials with real coefficients.

On a neighborhood of $\underline{\eta}$, $\mathbf{V_R} = \{P_\alpha(\eta) = 0\}$ for some α and $dP_\alpha(\underline{\tau}, \underline{\eta})$ is a nonzero real covector. The implicit function theorem implies that on a real neighborhood of $\underline{\eta}$, $\mathbf{V_R}$ is a real analytic manifold of codimension equal to 1. Thus $\underline{\eta}$ is a regular point of $\mathbf{V_R}$.

For the harder converse direction, suppose that $\underline{\eta}$ is a regular point of $\mathbf{V_R}$.

A first step is to show that $\theta = (1, 0, \ldots, 0)$ is not tangent to $\mathbf{V_R}$ at $\underline{\eta}$. This is equivalent to the fact that near $\underline{\tau}, \underline{\eta}$, $\mathbf{V_R}$ has a parameterization $\tau = \lambda(\xi)$ with $\lambda \in C^\omega$.

Denote by ν a conormal vector to $\mathbf{V_R}$ at $\underline{\eta}$. We need to show that

$$\langle \nu\,,\, (1, 0, \ldots, 0)\rangle \ \neq 0\,. \tag{1}$$

This follows from the algorithm in [R] for computing ν. We give an independent elementary proof and obtain, as a Corollary, a second proof of the result in [R].

The proof is by contradiction. If (1) were not true, changing linear coordinates ξ yields

$$\nu = (0, 1, 0, \ldots, 0)\,.$$

Then near $\underline{\tau}, \underline{\eta}$, $\mathbf{V_R}$ has an equation

$$\xi_1 \ = \ f(\tau, \xi')\,, \qquad \xi' := (\xi_2, \ldots, \xi_n)\,, \tag{2}$$

with

$$f \in C^\omega\,, \qquad f(\underline{\tau}, \underline{\xi}') = \underline{\xi}_1\,, \qquad \partial_{\tau,\xi'} f(\underline{\tau}, \underline{\xi}') = 0\,. \tag{3}$$

For ξ' fixed equal to $\underline{\xi}'$ expand f about $\tau = \underline{\tau}$,

$$f(\tau, \underline{\xi}') = \underline{\xi}_1 + a(\tau - \underline{\tau})^r + \text{higher order terms}\,, \qquad a \in \mathbf{R} \setminus 0\,.$$

The gradient condition in (3) implies that the integer $r \geq 2$. Solving (2) for τ as a function of ξ_1 shows that for ξ_1 near $\underline{\xi}_1$, there are r distinct complex roots $\tau \approx \big[(\xi_1 - \underline{\xi}_1)/a\big]^{1/r}$. Since $r \geq 2$, real values of ξ_1 near $\underline{\xi}_1$ with $(\xi_1 - \underline{\xi}_1)/a < 0$ yield nonreal solutions τ violating the hypothesis of hyperbolicity. This contradiction proves (1).

Thus, near $(\underline{\tau}, \underline{\xi})$, $\mathbf{V_R}$ is given by a real analytic equation $\tau = \lambda(\xi)$.

Dividing P by $P(1, 0, \ldots, 0)$ reduces to the case where this coefficient is equal to 1. Consider

$$P(\tau, \xi) \;=\; \tau^m + p_1(\xi)\tau^{m-1} + p_2(\xi)\tau^{m-2} + \cdots + p_m(\xi)$$

as a polynomial in τ depending on ξ. For $\xi = \underline{\xi}$, denote by k, the multiplicity of the root $\underline{\tau} = \lambda(\underline{\xi})$. For real ξ near $\underline{\xi}$, $P(\tau, \xi) = 0$ has exactly k roots near $\lambda(\underline{\xi})$. Hyperbolicity implies that they are all real. Since $\mathbf{V_R}$ has equation $\tau = \lambda(\xi)$ near $\underline{\xi}$ there is exactly one point $(\lambda(\xi), \xi)$ near $(\underline{\tau}, \underline{\xi})$ projecting to ξ. Therefore the k roots are all equal to $\lambda(\xi)$, so $\lambda(\xi)$ is a root of multiplicity exactly equal to k.

For ξ near $\underline{\xi}$, divide $P(\tau, \xi)$ by $\big(\tau - \lambda(\xi)\big)^k$,

$$P(\tau, \xi) \;=\; \big(\tau - \lambda(\xi)\big)^k \Big(\tau^{m-k} + a_1(\xi)\tau^{m-k-1} + \cdots + a_{m-k}(\xi)\Big) = \tag{4}$$

$$\Big(\tau^k + C_1^k \tau^{k-1}(-\lambda) + C_2^k \tau^{k-2}(-\lambda)^2 + \cdots + (-\lambda)^k\Big)\Big(\tau^{m-k} + a_1(\xi)\tau^{m-k-1} + \cdots + a_{m-k}(\xi)\Big)\,.$$

Equating coefficients of powers of τ yields the equations

$$p_1 = C_1^k(-\lambda) + a_1\,,$$
$$p_2 = C_2^k(-\lambda)^2 + C_1^k(-\lambda)a_1 + a_2\,,$$

and so on. Since λ and p_1 are real analytic, the first equality implies that a_1 is a real analytic function of ξ. Then, the second equality implies that a_2 is real analytic. By induction one finds that all the $a_j(\xi)$ are real analytic functions of ξ. Unique continuation implies that (4) holds on a complex neighborhood of $(\underline{\tau}, \underline{\xi})$.

Since the second factor on the right of (4) is nonzero at $(\underline{\tau}, \underline{\xi})$, it is nonzero on a complex neighborhood of this point. Therefore, on a complex neighborhood of $(\underline{\tau}, \underline{\xi})$, the complex variety $\mathbf{V_C}$ is given by the analytic equation $\tau = \lambda(\xi)$. Thus, $(\underline{\tau}, \underline{\xi})$ is a regular point of $\mathbf{V_C}$ and the proof is complete. ∎

Theorem. *If $\underline{\eta}$ is a regular point of the hyperbolic characteristic variety $\mathbf{V_R}$, there is an algebraic algorithm which determines the germ of $\mathbf{V_R}$ at $\underline{\eta}$ in terms of the germ of P at $\underline{\eta}$.*

Remarks. 1. Our proof shows more generally that at a regular point of a complex algebraic variety the germ of P determines the germ of $V_{\mathbf{C}}$. Thanks to the Theorem proved above this suffices for hyperbolic characteristic varieties. **2.** An independent computation of the tangent plane at $\underline{\eta}$ is given in [R]. **3.** The tangent plane at $\underline{\eta}$ gives the group velocity and thereby the transport operators of geometric optics. The second order terms in the germ of $\mathbf{V_R}$ yield the linear Schrödinger operator of diffractive geometric optics [DJMR].

Proof. Introduce coordinates $\eta = (\tau, \xi)$ as in the preceding proof. Then as in that proof, on a neighborhood of $(\underline{\tau}, \underline{\xi})$, the real and complex varieties have equation $\tau = \lambda(\xi)$ with real analytic λ and P factors

$$P = \big(\tau - \lambda(\xi)\big)^k\, q(\eta) \tag{5}$$

with real analytic q satisfying $q(\underline{\eta}) \neq 0$. What is required is an algorithm to compute the germ of λ at $\underline{\xi}$.

The factorization (5) implies that

$$P(\underline{\eta} + \eta) = O(|\eta|^k) \quad \text{and} \quad P(\underline{\eta} + \eta) \neq O(|\eta|^{k+1})\,.$$

Therefore, the Taylor expansion of P about $\underline{\eta}$ begins with a term of order k,

$$P(\underline{\eta} + \eta) \sim P_k(\eta) + P_{k+1}(\eta) \dots\,, \tag{6}$$

where the P_j are homogeneous polynomials of degree j. The finite set of polynomials on the right are computed algebraically by expanding the left hand side and collecting according to the power of η.

Write the Taylor expansions

$$\lambda(\underline{\xi} + \xi) - \lambda(\underline{\xi}) \sim \ell_1(\xi) + \ell_2(\xi) + \dots\,, \qquad q(\underline{\eta} + \eta) \sim q_0 + q_1(\eta) + q_2(\eta) + \dots\,, \tag{7}$$

with the same convention about the homogeneity of the polynomials ℓ_j and q_j. Then

$$(\tau - \lambda)(\underline{\eta} + \eta) \;\sim\; \mu_1(\eta) + \mu_2(\eta) + \dots\,, \quad \text{where,} \qquad \mu_1(\eta) = \tau - \ell_1(\xi)\,, \quad \mu_j(\eta) = -\ell_j(\xi) \;\text{ for }\; j \geq 2\,.$$

We give an algebraic algorithm determining the polynomials ℓ_j and q_j from the polynomials P_j.

The strategy is straight forward. Inject the expansions (7) in the right hand side of (5). Collecting according to powers of η yields identities relating the polynomials P_j, ℓ_j, q_j. It is then shown that these relations determine the ℓ_j and q_j from the P_j.

The first term begins the induction,

$$P_k \;=\; \mu_1^k\, q_0 \;=\; (\tau - \ell_1)^k\, q_0 \;=\; \tau^k q_0 - k\, q_0\, \ell_1(\xi)\, \tau^{k-1} + \cdots\,, \tag{8}$$

It yield the constant term q_0 and also ℓ_1 by the formulas

$$q_0 \;=\; \text{the coefficient of } \tau^k \text{ in } P_k(\tau, \xi)\,, \qquad \ell_1(\xi) \;=\; \frac{\text{the coefficient of } \tau^{k-1} \text{ in } P_k(\tau, \xi)}{-k\, q_0}\,. \tag{9}$$

The next term is typical of the inductive step,

$$P_{k+1} \;=\; \mu_1^k\, q_1 + k\, \mu_1^{k-1}\, \mu_2\, q_0 \;=\; (\tau - \ell_1)^{k-1}\Big((\tau - \ell_1(\xi))q_1 - k\, \ell_2(\xi)\, q_0\Big)\,. \tag{10}$$

It follows that the polynomial $(\tau - \ell_1)^{k-1}$ divides P_{k+1} and one can therefore compute

$$\frac{P_{k+1}(\tau, \xi)}{(\tau - \ell_1)^{k-1}} \;=\; (\tau - \ell_1(\xi))q_1 - k\, \ell_2(\xi)\, q_0\,. \tag{11}$$

Setting $\tau = \ell_1(\xi)$ yields a formula for ℓ_2,

$$-\ell_2(\xi) \;=\; \frac{P_{k+1}(\tau,\xi)}{(\tau-\ell_1)^{k-1}\,k\,q_0}\bigg|_{\tau=\ell_1(\xi)}\,. \tag{12}$$

Once ℓ_2 is known, one concludes from (11) that

$$\frac{P_{k+1}(\tau,\xi)}{(\tau-\ell_1)^{k-1}} + k\,\ell_2\,q_0 \;=\; \frac{P_{k+1}(\tau,\xi) + (\tau-\ell_1)^{k-1}\,k\,\ell_2\,q_0}{(\tau-\ell_1)^{k-1}}$$

is divisible by $\tau - \ell_1$ and one has the formula

$$q_1 \;=\; \frac{P_{k+1}(\tau,\xi) + (\tau-\ell_1)^{k-1}\,k\,\ell_2\,q_0}{(\tau-\ell_1)^{k}}\,. \tag{13}$$

The term of order $k+1$ has been used to determine ℓ_2 and q_1.

Suppose next that $j \geq 1$ and that $q_0, q_1, \ldots, q_{j-1}$ and $\ell_1, \ell_2, \ldots, \ell_j$ have been determined. We show how the term of order $k+j$ determines q_j and ℓ_{j+1}. Injecting (6) and (7) in (5) and then extracting the terms of order $k+j$ in η yields

$$P_{k+j} \;=\; \mu_1^k\,q_j + k\,\mu_1^{k-1}\mu_{j+1}\,q_0 + F(\mu_1,\ldots,\mu_j,q_0,\ldots,q_{j-1})\,, \tag{14}$$

where the last term is a polynomial in terms already determined. Thus the term F is known. Equation (14) shows that the polynomial $P_{k+j} - F$ is divisible by μ_1^{k-1} and

$$\frac{P_{k+j}-F}{\mu_1^{k-1}} \;=\; (\tau-\ell_1)\,q_j - k\,\ell_{j+1}\,q_0\,.$$

This yields the formula

$$-\ell_{j+1}(\xi) = \mu_{j+1}(\xi) \;=\; \frac{P_{k+1}-F}{\mu_1^{k-1}\,k\,q_0}\bigg|_{\tau=\ell_1(\xi)}\,. \tag{15}$$

This determined, (14) yields the formula

$$q_j \;=\; \frac{P_{k+1} - k\,\mu_1^{k-1}\mu_{j+1} - F}{\mu_1^k}\,. \tag{16}$$

This completes the inductive proof. ∎

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