Math 471. **Insert in §5.2.4. Comments on Matrix Norms.**

A matrix norm is an assignment of a nonnegative number $\|A\|$ to every $m \times n$ matrix. The assignment must have the following three properties.

1. $\|A\| = 0$ if and only if $A = 0$.
2. $\alpha\|A\| = |\alpha|\|A\|$ for all scalars α.
3. $\|A + B\| \le \|A\| + \|B\|$.

If A is an $m \times n$ matrix then for any $x \in \mathbf{R}^n$, $Ax \in \mathbf{R}^m$ so A defines a mapping from $\mathbf{R}^n$ to $\mathbf{R}^m$. If you have chosen a vector norm $||\ ||_{\mathbf{R}^m}$ for $\mathbf{R}^m$ and $||\ ||_{\mathbf{R}^n}$ for $\mathbf{R}^n$, then there is a naturally induced (or subordinate) matrix norm defined on the $m \times n$ matrices by

$$||A|| = \max_{||x||_{\mathbf{R}^n}=1} ||Ax||_{\mathbf{R}^m} = \max_{||x||_{\mathbf{R}^n}\le 1} ||Ax||_{\mathbf{R}^m} = \max_{x\neq 0} \frac{||Ax||_{\mathbf{R}^m}}{||x||_{\mathbf{R}^n}}. \tag{1}$$

The easy verification that these expressions are equal was presented in class. They show that the induced norm has the property that

$$||Ax||_{\mathbf{R}^m} \le ||A||\, ||x||_{\mathbf{R}^n},$$

and is the smallest number with this property. The induced norm is the "worst case expansion factor". Sometimes the induced norms are easy to compute.

Formula for the matrix norm induced by the vector norms $\|\ \|_\infty$. With this vector norm in both $\mathbf{R}^m$ and $\mathbf{R}^n$ we showed in class that the induced norm is given by the simple formula

$$\|A\|_\infty = \max_i \sum_j |a_{ij}|. \tag{2}$$

This is the maximum row sum of the matrix A. The fact that it is induced by the ∞-norm explains why it is denoted $||A||_\infty$ in the text. A computation of the same type is next given to compute a formula for the matrix norm induced by $||\ ||_1$ on both $\mathbf{R}^m$ and $\mathbf{R}^n$.

Formula for the matrix norm induced by the vector norms $\|\ \|_1$.

Compute

$$\begin{aligned}
||Ax||_1 &= \sum_i |(Ax)_i| \\
&= \sum_i \Big| \sum_j a_{ij}\, x_i \Big| \\
&\le \sum_i \sum_j |a_{ij}|\, |x_i| \\
&= \sum_j \sum_i |a_{ij}|\, |x_i| \\
&\le \max_j \Big\{ \sum_i |a_{ij}| \Big\} \sum_i |x_i|.
\end{aligned}$$

This shows that the induced norm satisfies

$$||A|| \leq \max_j \left\{ \sum_i |a_{ij}| \right\}. \tag{3}$$

On the other hand, choose k so that the k^{th} column sum is maximal,

$$\max_j \left\{ \sum_i |a_{ij}| \right\} = \sum_i |a_{ik}|.$$

Then with $x = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the one in the k^{th} coordinate we have $||x||_1 = 1$ and $Ax = (a_{1k}, a_{2k}, \ldots, a_{mk})$ is the k^{th} column of the matrix A. Then

$$||Ax||_1 = ||(a_{1k}, a_{2k}, \ldots, a_{mk})||_1 = \sum_i |a_{ik}| = \max_j \left\{ \sum_i |a_{ij}| \right\}$$

which shows that the induced norm satisfies

$$||A|| \geq \max_j \left\{ \sum_i |a_{ij}| \right\}. \tag{4}$$

Estimates (3) and (4) show that *the norm induced by the 1-norms is given by*

$$||A|| = \max_j \left\{ \sum_i |a_{ij}| \right\} \tag{5}$$

which is the maximal column sum of the matrix A. This computation explains why this is denoted $||A||_1$ in the text.

In contrast to the simple formulas above, the norm $||\ ||_2$ induced by the Euclidean norm is not easy to find. We illustrate this by computing one example.

Example. Compute exactly $||A||_2$ induced by the Euclidean norm on $\mathbf{R}^2$ when

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Solution. From the definition of the induced norm, one must find the maximum of

$$||Ax||_2^2 = (x_1 + x_2)^2 + x_2^2 = x_1^2 + 2x_2^2 + 2x_1 x_2 \tag{6}$$

subject to the constraint

$$||x||_2^2 = x_1^2 + x_2^2 = 1. \tag{7}$$

These norms are the Euclidean norms, since the matrix norm is that induced by the Euclidean norm. Using the constraint (7) in (6) we see that it is sufficient to maximize $x_2^2 + 2x_1x_2$ subject to the constraint (7). The method of Lagrange multipliers asserts that at the points (x_1, x_2) where the constrained extrema are achieved one must have

$$\nabla_x (x_2^2 + 2x_1 x_2) \parallel \nabla_x (x_1^2 + x_2^2).$$

This holds if and only if

$$0 = \det \begin{pmatrix} 2x_2 & 2x_2 + 2x_1 \\ 2x_1 & 2x_2 \end{pmatrix}$$

which yields

$$x_1(x_1 + x_2) = x_2^2 \,. \tag{8}$$

The clever work is now done. Equations (7) and (8) are two equations for the two unknowns x_1, x_2. Using equation (7) to eliminate x_2 from equation (8) yields

$$x_1\sqrt{1 - x_1^2} = 1 - 2x_1^2 \,.$$

Square to find $x_1^2(1 - x_1^2) = (1 - 2x_1^2)^2$. This is a quadratic equation

$$s(1 - s) = (1 - 2s)^2$$

for $s := x_1^2$. Simplifying yields $0 = 1 - 5s + 5s^2$ with solutions

$$x_1^2 = s_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{25 - 20}}{10} = \frac{5 \pm \sqrt{5}}{10} \,. \tag{9}$$

Then

$$x_2^2 = 1 - x_1^2 = 1 - s_\pm = \frac{10}{10} - \frac{5 \pm \sqrt{5}}{10} = \frac{5 \mp \sqrt{5}}{10} = s_\mp \,. \tag{10}$$

The function to be maximized is the right hand side of (6) and we have shown that one need only consider x_j satisfying (9) and (10). The maximum is achieved when $x_2 = \pm\sqrt{s_+}$ and $x_1 = \pm\sqrt{s_-}$ have the same sign and x_2 is larger. The minimum is obtained when the signs are the same and x_2 is smaller. The choice of opposing signs does not satisfy equation (8) so is not in competition. This spurious root came from squaring in the solution above.

The norm is the square root of the maximum of the right hand side of (6) which yields

$$||A||_2 = \left(s_- + 2s_+ + 2\sqrt{s_+ s_-}\right)^{1/2} \approx 1.680 \,.$$

The right hand side is matlab's answer norm(A,2).

Problem 5.2.12. If $\mathbf{R}^n$ is given the 1-norm and $\mathbf{R}^m$ is given the ∞-norm show that the induced norm on matrices is given by the formula

$$||A||_{1,\infty} = \max_{i,j} \left\{ |a_{ij}| \right\} . \tag{11}$$

Problem 5.2.13. Show that one gets exactly the same formula (11) for the induced norm $||A||_{\infty,1}$ when $\mathbf{R}^n$ is given the ∞-norm and $\mathbf{R}^m$ is given the 1-norm.

Problem 5.2.14. Compute exactly $||B||_2$ induced by the Euclidean norm on $\mathbf{R}^2$ when

$$B = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} .$$

You can check your answer with the Matlab command norm(B,2). **Discussion.** The norms induced by the 1-norm and ∞-norm are given by formulas (2) and (5), $||B||_1 = 4$ and $||B||_\infty = 3$.