Global Solvability of the Maxwell-Bloch Equations from Nonlinear Optics

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§1. Introduction.

The nonlinear self-focusing of optical beams is a well known and abundantly documented phenomenon [M, Sh]. The mechanism is simple. If the speed of propagation of electromagnetic waves decreases as a function of the intensity, then ray tracing suggests that a planar wavefront with intensity which is large at the center and decreases away from the center will propagate so that the center lags behind the edges. This initial curvature translates into focussing after a finite period of time.

Interest in this phenomenon has been recently renewed with the advent of ultrashort and ultraintense pulses [Ro]. For ultraintense pulses self-focusing is undesirable as it often leads to the extinction of the beam and to the destruction of very costly optical devices. In order to better understand focussing, with one goal being avoidance, good models are needed for the regions of space time where laser beams focus.

The usual analysis of self-focusing uses the nonlinear Schrödinger equation (NLS). This arises from more fundamental field equations by the slowly varying envelope approximation. One seeks an approximate solution in the form of a slowly varying field envelope times a rapidly oscillating term with linear phase function. A medium with an

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instantaneous cubic nonlinear response yields the NLS with cubic nonlinearity. This model predicts the instability of plane waves to transverse oscillations, and thereby the onset of self-focusing. It also lead to the prediction that it is possible for the self-focusing effect from the nonlinearity to exactly balance the diffusive effects from the Schrödinger equation leading to self-trapping [CGT]. It was soon observed both experimentally and theoretically that this trapping is not stable to perturbations in the two transverse dimensions. These early successes of the NLS established it and near relatives as the standard models for laser beam propagation.

In spite of this acceptance, this approximation remained little analysed for many years. The slowly varying envelope approximation has recently been justified by [DR], taking advantage of many analytical advances in the work of Joly, Metivier, Rauch, and Gues on nonlinear geometric optics. The approximation is the leading term in an asymptotic expansion valid as the wavelength tends to zero. In regions where the solution of the Schrödinger equation is regular, it gives a good approximation.

In the case of three dimensional space, solutions of the NLS equation often develope singularities in finite time [Tal], [K], [G]. The standard interpretation is that this is symptomatic of catastrophic self-focusing. Our main result asserts that in the case of one fundamental model, the Maxwell-Bloch system, solutions remain regular for all $t \geq 0$. They do not develope singularities in finite time. The singularity formation in the Schrödinger equation is symptomatic of the failure of the approximation. Though the approximate solution tends to infinity, the exact solution remains perfectly finite. The error in the approximation tends to infinity.

This is entirely analogous to the singularities in the amplitudes in linear geometric optics as one approaches caustics. Solutions of the linear Maxwell Equations do not develope singularities, but, the asymptotic approximations do. To complete the picture for the nonlinear optics, one would need a more accurate asymptotic description near singularities of the Schrödinger equation. This appears to us to be a very difficult problem.

Near such singular points the behavior of the solution of the nonlinear Maxwell equations is not well understood. In fact different fundamental field equations with the same NLS approximation, may behave differently near the singular points. Thus an understanding of these behaviors can help in selecting correct models.

One expects that for extremely high fields, that additional physical effects must be included, and that these should remove the singularity. Many attempts to modify the NLS model, none with much justification or success, have been suggested with this goal in mind. This includes addings to the NLS equation an *ad hoc* saturation of the nonlinear index [DM], multiphoton absorption [DLP], spatial and temporal nonparaxial terms [FF], [Rot2], or group velocity dispersion [Rot1], [MP], [LMN]. Only for the saturated nonlinearity is there a rigorous proof that the singularity is eliminated.

In this paper, we study the more fundamental Maxwell equations. The medium is modelled as a gas of quantum mechanical systems with a finite number of energy levels. Such models, called Maxwell-Bloch (MB) systems, are often used to study the phenomena of resonance. We use it to model wave propagation. The electromagnetic field is modelled classically, while the medium is quantum mechanical. We prove rigorously that the solutions of this model remain regular provided that the initial condition is smooth.

The MB system produces a focusing or defocusing NLS equation as an asymptotic expansion, depending on the details of the finite state quantum system [D], [BC]. For example, a three level transition yields a focusing NLS equation under classical conditions [D]. Another example is provided by the low lying energy levels for helium which show also that this medium is focusing in the low frequency limit [BL].

For simplicity, we develop in this paper the simplest MB equation, that with two levels. The results remains valid for a finite number of levels independent of the focusing or defocusing behavior of approximating NLS.

The unknowns in the Maxwell-Bloch system include three 3-vector fields; the electric and magnetic fields $E(t,x)$ and $B(t,x)$, and the polarization $P(t,x)$ of the medium. A scalar field $N(t,x)$ describes the difference betweeen the number of electrons in the excited state and the ground state per unit volume. The number N_0 is the equilibrium value of N. In dimensionless notation the system is (see for example [PP]),

$$\partial_t B + curl\, E = 0 \tag{1.1}$$

$$\partial_t\, E - curl\, B = -\partial_t P \tag{1.2}$$

$$\partial_t^2 P + \partial_t P / T_2 + \Omega^2 P = c_1 N E \tag{1.3}$$

$$\partial_t N + (N - N_0)/T_1 = -c_2 \langle \partial_t P, E \rangle . \tag{1.4}$$

The nonegative scalars c_1, and c_2 determine the strength of the coupling between the equations. The scalars T_1 and T_2 are called the inversion population lifetime and the homogeneous dephasing time. Ω is a resonant frequency which often corresponds to the principal optical spectral line of the medium.

Taking the divergence of (1.1) and (1.2) implies that

$$\partial_t\, div\,(E + P) = \partial_t\, div\,(B) = 0. \tag{1.5}$$

The physically relevant solutions are those which satisfy

$$div(E + P) = div(B) = 0. \qquad (1.6).$$

Thanks to (1.5) this holds as soon as it holds at t=0, so (1.6) is only a linear constraint on the initial data.

Let

$$U(t, x) := (E(t, x)\,, B(t, x)\,, P(t, x)\,, \partial_t P(t, x)\,, N(t, x) - N_0). \tag{1.7}$$

Equation 1.3 is written as a system for the pair (P, Q) with $Q := \partial_t P$,

$$\partial_t\, P = Q, \qquad \partial_t\, Q = -Q/T_2 - \Omega^2 P + c_1 N E.$$

The Maxwell Bloch system then takes the form of a semilinear symmetric hyperbolic system for U,

$$\partial_t\, U = \sum_{1 \le j \le 3} A_j \partial_j U + F(U)$$

where $\partial_j := \partial / \partial x_j$, the A_j are symmetric matrices, and $F : \mathbf{R}^{13} \to \mathbf{R}^{13}$ is a polynomial of degree two which vanishes at the origin. The standard local existence theorem for semilinear hyperbolic equations yields the following result.

Local Existence Theorem. *If $s > 3/2$ and $U(0, .) \in H^s(\mathbf{R}^3)$, then there is a $T_* \in]0, \infty]$ and a unique $U \in C([0, T_*[: H^s(\mathbf{R}^3))$ which satisfies the Maxwell Bloch system and attains* *these initial values. The solution depends continuously on the initial data in the sense that if $\varphi \in H^s(\mathbf{R}^3)$ and $T < T_*(\varphi)$, then there is an $H^s(\mathbf{R}^3)$ neighborhood $\mathcal{O}$ of φ such that $T_*(\psi) > T$ for all $\psi \in \mathcal{O}$ and the map from initial data to solution is continuous from $\mathcal{O}$ to $C([0,T] : H^s(\mathbf{R}^3))$. There is a $c > 0$ so that $T_* > c(s)/(1 + \|U(0\|_{H^s(\mathbf{R}^3)})$. The values of U at $(\underline{t}, \underline{x})$ depend only on the values of the initial data on the ball of radius $\underline{t}$ with center $\underline{x}$.*

Elements of the proof. For any $s \in \mathbf{R}$ the operator $\sum_{1\le j\le 3} A_j\partial_j U$ is antiselfadjoint on $H^s(\mathbf{R}^3)$. Schauder's lemma shows that for $s > 3/2$, the nonlinear term $F(U)$ is a locally Lipshitzean function on $H^s(\mathbf{R}^3)$. All the assertions of the lemma then follow by straight forward Picard iteration as in [Re, Thm. 1]. $\diamondsuit$

It follows that to prove global solvability for data $U(0,.)$ belonging to H^s for $s \ge 2$ it suffices to prove the following *a priori* estimate. For any $T \in]0,\infty[$ and any $M > 0$ there is a constant $C(T,M)$ such that if $\underline{t} \in]0,T]$ and U is a smooth compactly supported solution on $[0,\underline{t}] \times \mathbf{R}^3$ such that

$$\|U(0)\|_{H^2(\mathbf{R}^3)} \le M\,, \tag{1.8}$$

then

$$\|U(t,.)\|_{H^2(\mathbf{R}^3)} \le C(T,M), \quad \text{for} \qquad 0 \le t \le \underline{t}. \tag{1.9}$$

To prove that 1.9 is sufficient, one must show that if $U(0) \in H^2(\mathbf{R}^3)$ and $T > 0$, then $T_* > T$. Choose $U_n(0) \in C_0^\infty(\mathbf{R}^3)$ converging to $U(0)$ in $H^2(\mathbf{R}^3)$. Let $M = sup\,||U_n(0)||_{H^2(\mathbf{R}^3)}$ and $\delta = c(2)/(1 + C(T,M))$ where $c(s)$ comes from the lower bound on the lifetime in the Local Existence Theorem. Then there are solutions $U_n \in C([0,\delta] : H^2(\mathbf{R}^3))$. By uniqueness this is the $C([0,\delta] : H^s(\mathbf{R}^3))$ for all $s \ge 2$ and it follows that U_n belongs to $C^\infty([0,\delta] \times \mathbf{R}^3)$. Finite speed shows that $U_n \in C^\infty_{(0)}([0,\delta] \times \mathbf{R}^3)$. Therefore 1.9 implies that the the $C([0,\delta] : H^2(\mathbf{R}^3))$ norm is at most $C(T,M)$. Restarting the process at $t = \delta$ constructs U_n in $C^\infty_{(0)}([0,2\delta] \times \mathbf{R}^3)$ with $C([0,2\delta] : H^2(\mathbf{R}^3))$ norm is at most $C(T,M)$. Continuing yields U_n in $C^\infty_{(0)}([0,T] \times \mathbf{R}^3)$ with $C([0,T] : H^2(\mathbf{R}^3))$ norm is at most $C(T,M)$.

Choosing a weak star convergent subsequence constructs a solution U belonging to $L^\infty([0,T] : H^2(\mathbf{R}^3))$. A simple L^2 uniqueness proof shows that U is *the* solution and therefore belongs to $C([0,T] : H^s(\mathbf{R}^3))$. Thus $T_* > T$

In the next section we prove 1.9 for initial data which satisfy the divergence condition 1.6.

Main Theorem. *If $s \geq 2$ and the initial data $U(0,.) \in H^s(\mathbf{R}^3)$ satisfy 1.6, then $T_* = \infty$. That is, there is a unique global solution belonging to $C([0,\infty[: H^s(\mathbf{R}^3))$.*

The proof proceeds by a sequence of *a priori* estimates finally constructing the function $C(T,M)$. The estimates derived in §2 through §5 do not use the divergence equation 1.6, and depend only on T and

$$m := \|U(0)\|_{H^1(\mathbf{R}^3)} + \|P(0)\,,\partial_t P(0)\,,N(0)\|_{L^\infty(\mathbf{R}^3)}. \tag{1.10}$$

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§2. The Fundamental L^2 Estimate.

The law of energy balance is demonstrated by taking the scalar product of (1.1) with $N_0 B$, (1.2) with $N_0 E$, and (1.3) with $\partial_t P/c_1$. Equation (1.4) is multiplied by $(N-N_0)/c_2$ and the resulting expressions are added. The products of the right hand sides with the multipliers is

$$-\partial_t P\,.\,|N_0|\,E - c_1 N\,E\,.\,\partial_t P/c_1 + c_2\,\partial_t P\,.\,E\,(N-N_0)/c_2 = 0\,.$$

This cancellation and a similar one presented in the next step are the keys to the existence proof. Using the vector identity $B.curl\, E - E.curl\, B = div(E \wedge B)$ yields the dissipation law

$$\partial_t e + div(N_0 E \wedge B) = -\frac{|\partial_t P|^2}{T_2 c_1} - \frac{(N - N_0)^2}{T_1 c_2}, \tag{2.1}$$

where

$$2e(t,x) := N_0|E|^2 + N_0|B|^2 + \frac{|\partial_t P|^2 + \Omega^2 P^2}{c_1} + \frac{(N - N_0)^2}{c_2}. \tag{2.2}$$

The consequeces of 2.1 are our only estimates which do not grow in time.

Integrating the dissipation law over the truncated cone $|x| \le R - t,\ 0 < t < T < R$ shows that the Maxwell Bloch system has speed of propagation at most one. Letting R tend to infinity shows that solutions U which are suitably small at infinity satisfy the energy dissipation identity,

$$\int_{\mathbf{R}^3} e(t,x)\, dx + \int_0^t \int_{\mathbf{R}^3} \frac{|\partial_t P|^2}{T_2 c_1} + \frac{(N - N_0)^2}{(T_1 c_2)}\, dx\, dt = \int_{\mathbf{R}^3} e(0,x)\, dx \tag{2.3}$$

In particular, $\int_{\mathbf{R}^3} e(t,x)\, dx$ is a decreasing function of time .

§3. An L^∞ estimate for P, $\partial_t P$, and N.

The key to proving global solvability and global regularity is to show that solutions are bounded on $[0,T] \times \mathbf{R}^3$. In this section we prove such boundedness for $P, Q := \partial_t P$, and N.

Take the scalar product of (1.3) with $2\partial_t P/c_1$ and multiply (1.4) by $2N/c_2$. Add the resulting expressions. The cubic terms from the right hand sides cancel exactly. This leaves

$$\partial_t \left(\frac{|\partial_t P|^2 + \Omega^2 |P|^2}{c_1} + \frac{N^2}{c_2} \right) + \frac{2|\partial_t P|^2}{c_1 T_2} + \frac{2N^2}{c_2 T_1} = \frac{2 N N_0}{c_2 T_1}. \tag{3.1}$$

The right hand side is less than or equal to

$$\frac{1}{c_2 T_1} \left(2N^2 + N_0^2/2 \right).$$

The first term is absorbed by last term on the left hand side of (3.1). The second term is independent of time. Integrating the differential inequality shows that for all t, x

with $t \geq 0$,

$$\frac{|\partial_t P(t,x)|^2 + \Omega^2 |P(t,x)|^2}{c_1} + \frac{N(t,x)^2}{c_2} \leq \frac{|\partial_t P(0,x)|^2 + \Omega^2 |P(0,x)|^2}{c_1} + \frac{N(0,x)^2}{c_2} + \frac{t N_0^2}{2 c_2 T_1}\,. \tag{3.2}$$

This gives a pointwise bound on P, $\partial_t P$ and N which grows at most as $\sqrt{t}$. It is worth noting that one could get pointwise bounds for negative time which grow like $\exp(c\sqrt{|t|})$

§4. H^1 Estimates.

The L^∞ bounds for N and $\partial_t P$ control one of the factors in each of the quadratic nonlinear terms in the Maxwell Bloch system. This together with the same cancellations which gave us the L^2 bounds of §2 suffice to derive H^1 bounds.

Denote by ∂_μ partial differentiation with repect to t or x_j, $1 \leq j \leq 3$ depending on whether $\mu = 0$ or $1 \leq \mu \leq 3$. Apply ∂_μ to each of the equations (1.1) to (1.4). Then multiply the differentiated equation by ∂_μ applied to the corresponding multiplier from §2. Add the resulting expressions. The cubic terms in the sum are equal to

$$\partial_\mu(NE).\, \partial_\mu \partial_t P - (\partial_\mu N)\, \partial_\mu(\partial_t P.E).$$

The key is that the two terms in this difference with E undifferentiated cancel. The other terms have a factor of N or $\partial_t P$ for which we have an L^∞ estimate. For $t \geq 0$ let

$$Y(t) := \sup_{x \in \mathbf{R}^3} \big(\text{right hand side of (3.2)}\big)^{\frac{1}{2}}. \tag{4.1}$$

Let

$$2e_1(t,x) := \sum_{\mu=0}^{3} N_0 |\partial_\mu E|^2 + N_0 |\partial_\mu B|^2 + \frac{|\partial_\mu \partial_t P|^2 + \Omega^2 |\partial_\mu P|^2}{c_1} + \frac{|\partial_\mu (N - N_0)|^2}{c_2}. \tag{4.2}$$

The above computation shows that

$$|\, \partial_t \int_{\mathbf{R}^3} e_1(t,x)\, dx \,| \leq c Y(t) \int_{\mathbf{R}^3} e_1(t,x)\, dx. \tag{4.3}$$

Let

$$Y_1(t) := \int_{\mathbf{R}^3} e_1(t,x)\, dx\,. \tag{4.4}$$

Then (4.3) implies that for $t \geq 0$,

$$Y_1(t) \leq Z(t)\,, \tag{4.5}$$

where $Z(t)$ defined and smooth on $[0,\infty[$ is the solution of

$$\frac{dZ(t)}{dt} = cY(t)Z(t) \qquad Z(0) = Y_1(0)\,. \tag{4.6}$$

§5. C^α bound for the divergence free part of E.

The Fourier transform of the divergence and curl of E are equal to $\langle \xi, \hat{E}\rangle$ and $\xi \wedge \hat{E}$ respectively. Thus, decomposing $\hat{E}(\xi)$ into the part parallel and orthogonal to ξ is the $L^2(\mathbf{R}^3)$ orthogonal projection on the irrotational and divergence free parts.

Define a smooth projection valued function on $\mathbf{R}^3_\xi \setminus 0$ by

$$\Pi(\xi)v = v - \xi\langle\xi, v\rangle/\langle\xi,\xi\rangle. \tag{5.1}$$

Then $\Pi(D)$ defined by

$$\Pi(D)E := \mathcal{F}^{-1}\big(\Pi(\xi)(\mathcal{F}E)(\xi)\big) \qquad (5.2).$$

is a convolution operator which is a classical singular integral operator of degree zero. $\Pi(D)$ is the L^2 orthogonal projection onto the divergence free vector fields along the irrotational fields.

For any vector v and any ξ

$$\xi \wedge (\xi \wedge v) = -|\xi|^2\Pi(\xi)v.$$

It follows that

$$curl\, curl\, \Pi(D) = \Pi(D)\Delta, \qquad \text{and} \qquad curl\, curl\, (I - \Pi(D)) = 0. \tag{5.3}$$

Taking the time derivative of (1.2) and then using (1.1) to eliminate B yields

$$E_{tt} - curl\, curl\, E = -P_{tt}. \tag{5.4}$$

Applying $\Pi(D)$ and using (1.3) yields

$$(\partial_{tt} - \Delta)\Pi(D)\,E = -\Pi(D)\,(c_1 NE - \Omega^2 P - \partial_t P/T_2) \tag{5.5}$$

The estimates of §4 imply that the P and $\partial_t P$ terms on the right are bounded in the continuous functions with values in H^1. The next result shows that we can bound $N(t)E(t)$ in $H^s(\mathbf{R}^3)$ for any $s < 2/3$.

Lemma.. *For any $s < 2/3$, pointwise multiplication of functions defines a continuous bilinear map of $(L^\infty \cap H^1)(\mathbf{R}^3) \times H^1(\mathbf{R}^3)$ to $H^s(\mathbf{R}^3)$.*

Proof. The proof is by two interpolations. Sobolev's Lemma implies that if $\rho > 3/2$ then multiplication is continuous from $H^\rho(\mathbf{R}^3) \times H^0(\mathbf{R}^3)$ to $H^0(\mathbf{R}^3)$. Schauder's Lemma implies continuity from $H^\rho(\mathbf{R}^3) \times H^\rho(\mathbf{R}^3)$ to $H^\rho(\mathbf{R}^3)$. Interpolation proves continuity from $H^\rho(\mathbf{R}^3) \times H^\sigma(\mathbf{R}^3)$ to $H^\sigma(\mathbf{R}^3)$ for all $0 \le \sigma \le \rho$.

Let $\mathbf{X} := (L^\infty \cap H^1)(\mathbf{R}^3)$. Taking $\sigma = 1$ shows that multiplication is continuous from $\mathbf{X} \times H^\rho$ to H^1.

Since $\mathbf{X} \subset L^\infty$ it follows that multiplication is continuous from $\mathbf{X} \times H^0(\mathbf{R}^3)$ to $H^0(\mathbf{R}^3)$.

Interpolating between these two results with interpolation parameter $\theta = 1/\rho$ shows that multiplication is continuous from $\mathbf{X} \times H^1(\mathbf{R}^3)$ to $H^{1/\rho}(\mathbf{R}^3)$. Since this is valid for all $\rho > 3/2$, the proof is complete. $\diamondsuit$

Given the $L^\infty \cap H^1$ bounds for N and the H^1 bounds for E, it follows that EN and therefore $\Pi(D)(EN)$ are bounded in the space of continuous functions of t with values in $H^s(\mathbf{R}^3)$ for any $s < 2/3$.

Applying standard estimates for D'Alembert's wave equation (see for example [Ra, §3.7]) to 5.5, bounds $\Pi(D)(EN)$ as a continuous function with values in $H^{s+1}(\mathbf{R}^3)$ and $\partial_t \Pi(D)(EN)$ as a continuous function with values in $H^s(\mathbf{R}^3)$.

Sobolev's lemma shows that for $1/2 < s < 1$, $H^{s+1}(\mathbf{R}^3) \subset C^{s-1/2}(\mathbf{R}^3)$. The limiting value $s = 2/3$ yields $C^{1/6}$. Thus for $0 < \alpha < 1/6$, one has a bound

$$\|\Pi(D)E(t)\|_{C^\alpha(\mathbf{R}^3)} \le C(T, m, \alpha), \quad \text{for} \ \ 0 \le t \le \underline{t} \le T, \tag{5.6}$$

where m is defined in 1.10.

§6. L^∞ estimate for the irrotational part of E.

To bound the sup norm of the irrotational part $(I - \Pi(D))E$ of E, we use the divergence equation 1.6 and second derivatives of U. The second derivatives have not yet been estimated. A Gronwall argument in the next section completes the demonstration.

Equation 1.6 implies that

$$(I - \Pi(D)\,)\,E = -\big(I - \Pi(D)\,\big)\,P\,. \tag{6.1}$$

The right hand side is an operator of order zero applied to the bounded function P. Unfortunately such operators do not map bounded functions to bounded functions. An observation, dating at least to Yudovich [Y], is that a logarithmic correction is a partial remedy. Applying the version given in Proposition B.1.A of [T] yields

$$\|(I - \Pi(D))E(t)\|_{L^\infty(\mathbf{R}^3)} \le c\|P(t)\|_{L^\infty(\mathbf{R}^3)}\, \log\big(2 + \|P(t)\|_{H^2(\mathbf{R}^3)}\big)\,. \tag{6.2}$$

Combined with the estimate of the previous section one has for any solution on $[0, \underline{t}]$ and $t \le \underline{t}$,

$$\|E(t)\|_{L^\infty(\mathbf{R}^3)} \le c(m, T) + (\text{right hand side of 6.2}). \tag{6.3}$$

§7. The final H^2 estimate.

In this section the second derivatives of a solution on $[0, \underline{t}]$ with $\underline{t} \le T$ are estimated by an energy argument related to that used for the L^2 and H^1 estimates. A subtlety is that the differential inequality which we derive is logarithmically superlinear because of 6.2 and 6.3.

For each μ, ν, apply $\partial_\mu \partial_\nu$ to each of the equations 1.1 to 1.4. Multiply the resulting equations by $\partial_\mu \partial_\nu$ applied to the corresponding multiplier from §2. Add the resulting expressions. The contribution of the right hand sides is equal to

$$-(\partial_\mu \partial_\nu \partial_t P).\partial_\mu \partial_\nu N_0 E + (\partial_\mu \partial_\nu (NE)).\partial_\mu \partial_\nu \partial_t P - \partial_\mu \partial_\nu \langle \partial_t P, E\rangle.\partial_\mu \partial_\nu (N - N_0)\,.$$

There are four terms. The two quadratic terms cancel.

Let

$$2e_{\mu\nu}(t,x) := N_0|\partial_\mu\partial_\nu E(t,x)|^2 + N_0|\partial_\mu\partial_\nu B(t,x)|^2 + \frac{|\partial_\mu\partial_\nu\partial_t P(t,x)|^2 + \Omega^2|\partial_\mu\partial_\nu P(t,x)|^2}{c_1} + \frac{|\partial_\mu\partial_\nu(N(t,x)-N_0)|^2}{c_2}\,. \tag{7.1}$$

The above computation yields a balance law for second derivatives

$$\partial_t(\partial_\mu\partial_\nu e) + div(N_0\partial_\mu\partial_\nu E\,\wedge\,\partial_\mu\partial_\nu B) + \frac{|\partial_\mu\partial_\nu\partial_t P|^2}{T_2c_1} + \frac{(\partial_\mu\partial_\nu N)^2}{T_1c_2} = (\partial_\mu\partial_\nu(NE)).\,\partial_\mu\partial_\nu\partial_t P - (\partial_\mu\partial_\nu N)\,\partial_\mu\partial_\nu(\partial_t P.E)\,. \tag{7.2}$$

Let

$$Y_2(t) := \int_{\mathbf{R}^3} e(t,x) + e_1(t,x) + \sum_{\mu,\nu} e_{\mu\nu}(t,x)\ dx\ . \tag{7.3}$$

Adding 2.3, 4.4, and the result of integrating 7.2 over $\mathbf{R}^3$ yields

$$Y_2'(t) \le\ c\|U(t)\|^2_{H^2(\mathbf{R}^3)} + \sum_{\mu,\nu}\int_{\mathbf{R}^3}(\partial_\mu\partial_\nu(NE)).\,\partial_\mu\partial_\nu\partial_t P - (\partial_\mu\partial_\nu N)\,\partial_\mu\partial_\nu(\partial_t P.E)\,dx\,. \tag{7.4}$$

To estimate the integral on the right in 7.4, note that from the derivatives of the products there are two types of terms, those for which there is a factor which is not differentiated and those for which two factors are differentiated one time each. T Terms of the first type are dominated by

$$c\|(E,N,\partial_t P)(t)\|_{L^\infty(\mathbf{R}^3)}\|U(t)\|^2_{H^2(\mathbf{R}^3)}. \tag{7.5}$$

Terms of the second type are dominated by

$$c\|\partial(E,N,\partial_t P)(t)\|^2_{L^4(\mathbf{R}^3)}\|U(t)\|_{H^2(\mathbf{R}^3)}. \tag{7.6}$$

The interpolation inequality

$$\|\partial(E,N,\partial_t P)(t)\|^2_{L^4(\mathbf{R}^3)} \le c\|(E,N,\partial_t P)(t)\|_{L^\infty(\mathbf{R}^3)}\|(E,N,\partial_t P)(t)\|_{H^2(\mathbf{R}^3)}.$$

shows that 7.6 is also dominated by 7.5.

Next, the estimates 3.2 and 6.3 imply that

$$\|(E,N,\partial_t P)(t)\|_{L^\infty(\mathbf{R}^3)} \le c(m,T)\,\log\left(2+\|U(t)\|_{H^2(\mathbf{R}^3)}\right). \tag{7.7}$$

Using these estimates in 7.4 yields a differential inequality for $0 \le t \le \underline{t} \le T$,

$$Y_2'(t) \le c(m,T)\,\|U(t)\|^2_{H^2(\mathbf{R}^3)}\, log(2 + \|U(t)\|_{H^2(\mathbf{R}^3)})\,. \tag{7.8}$$

Finally note that Y_2 is larger than a constant times the square of the H^2 norm of $U(t)$ so one has

$$Y_2'(t) \le c(m,T)\,Y_2(t)\,\log(2 + Y_2(t)\,)\,. \tag{7.9}$$

Thus for $0 \le t \le \underline{t} \le T$,

$$Y_2(t) \le Z_2(t), \tag{7.10}$$

where $Z_2(t)$ is the solution of the globally solvable initial value problem

$$Z_2'(t) = c(m,T)\,\log(2 + Z_2(t))\,Z_2(t), \qquad Z_2(0) = Y_2(0). \tag{7.11}$$

This completes the proof of the estimate 1.9 and therefore the proof of the Main Theorem. $\diamondsuit$

The bound for the second derivative grows very rapidly with time, in fact, faster than a multiple exponential. We do not know whether the solutions actually grow anywhere near this fast.

§8. Global H^1 solutions.

Using the the fact that the estimates of §2 and §3 depend only on m defined in 1.10, it is not hard to prove the following result about global weak solutions.

Theorem. *For arbitrary $U(0) \in H^1(\mathbf{R}^3)$ such that $(N(0), P(0), \partial_t P(0)) \in L^\infty(\mathbf{R}^3)$, there is a unique solution such that for all $T > 0$, $U \in L^\infty([0,T] : H^1(\mathbf{R}^3))$ and $(N, P, \partial_t P) \in L^\infty([0,T] \times \mathbf{R}^3)$. The solution satisfies the energy dissipation identity 2.3, and, has speed of propagation at most one.*

Remarks. 1. If $U(0)$ is in L^∞, the estimates in §5 suffice to show that $\Pi(D)E$ is bounded on $[0,T] \times \mathbf{R}^3$. A similar argument shows that B is bounded. The only part of U for which sup norm bounds are not known is the irrotational part, $(I - \Pi(D))E$, of E.

2. Since there is an L^2 dissipation law with a natural physical interpretation, and since the nonlinearities are quadratic, it would be natural to seek weak solutions with $U(0)$ square integrable. This remains an open problem.

Proof of Theorem. The solutions are constructed as limits of approximate solutions U_ε whose initial data are equal to $J_\varepsilon U(0)$ where J_ε is a standard spatial mollifier

$$J_\varepsilon(\varphi) := \int_{\mathbf{R}^3} j(y)\varphi(x-\varepsilon y)\,dy, \quad \text{where,} \quad j \in C_0^\infty(\mathbf{R}^3), \quad \int_{\mathbf{R}^3} j(y)\,dy = 1.$$

The estimates of sections 2-4 show that the approximate solutions are bounded in the continuous funtions with values in H^1 and that $(N_\varepsilon, P_\varepsilon, \partial_t P_\varepsilon)$ is bounded in $L^\infty([0,T]\times\mathbf{R}^3)$.

Since each of the quadratic nonlinear terms has one bounded factor, the differential equations 1.1 through 1.4 show that the family $\partial_t U_\varepsilon$ is bounded in the continuous functions of time with values in $L^2(\mathbf{R}^3)$.

Combining the finite domain of dependence and the local compactness of H^1 in L^2 we can apply the Arzela-Ascoli theorem to choose a subsequence strongly convergent in $C([0,\infty[\,: L^2(\mathbf{R}^3)\,)$ to a limit U.

The strong convergence shows that we can pass to the limit in the nonlinear terms of the Maxwell Bloch system since they are quadratic. Similarly one can pass to the limit in the energy dissipation law. Thus U satisfies the system and the law.

The L^2 convergence and L^∞ boundedness imply that $(N_\varepsilon, P_\varepsilon, \partial_t P_\varepsilon)$ converges weak star in $L^\infty([0,T]\times\mathbf{R}^3)$. Similarly the solution constructed is in $L^\infty([0,T]\,:\,H^1(\mathbf{R}^3)\,)$. Thus

$$U \in L^\infty([0,T]\times\mathbf{R}^3) \cap L^\infty([0,T]:H^1(\mathbf{R}^3)\,)\,.$$

This completes the existence proof.

The proof of uniqueness depends on the cancellations leading to the L^2 estimate in §2. Suppose that one had two solutions, indicated by the superscripts 1 and 2. Subtract the equations 1.1 to 1.4 for the two solutions. Multiply the equations by $N_0(B^1-B^2)$, $N_0(E^1-E^2)$, $(\partial_t P^1 - \partial_t P^2)/c_1$, and $(N^1-N^2)/c_2$. The contributions of the right hand sides is

$$-(\partial_t P^1 - \partial_t P^2).N_0(E^1 - E^2)$$
$$+(N^1E^1 - N^2E^2).(\partial_t P^1 - \partial_t P^2) - (\partial_t P^1.E^1 - \partial_t P^2.E^2)(N^1 - N^2). \tag{8.1}$$

The first product is quadratic. Of the eight cubic terms the two terms which only have superscripts 1 cancel. Similarly the two terms with only superscripts 2.

Let

$$\begin{aligned} 2f(t,x) &:= N_0|E^1 - E^2|^2 + N_0|B^1 - B^2|^2 + \\ &\quad + \frac{|\partial_t(P^1 - P^2)|^2 + \Omega^2|P^1 - P^2|^2}{c_1} + \frac{(N^1 - N^2)^2}{c_2}. \end{aligned} \tag{8.2}$$

Then the quadratic terms in 8.1 are bounded by $cf(t,x)$. The above cancellation leaves four cross terms. Using the divergence identity from section 2, one finds a balance law

$$\begin{aligned} &\partial_t f(t,x) + q(t,x) + div(N_0(E^1 - E^2) \wedge (B^1 - B^2)) \\ &\quad \le cf(t,x) - \partial_t P^1 N^2.E^2 - \partial_t P^2 N^1.E^1 + N^1 \partial_t P^2.E^2 + N^2 \partial_t P^1.E^1, \end{aligned} \tag{8.3}$$

where

$$q(t,x) := \frac{|\partial_t P^1 - \partial_t P^2|^2}{T_2 c_1} + \frac{(N^1 - N^2)^2}{T_1 c_2} \ge 0\,.$$

The last four terms in 8.3 simplify to

$$(E^1 - E^2).(\partial_t P^2 N^1 - \partial_t P^1 N^2) = (E^1 - E^2).\Big(\partial_t P^2(N^1 - N^2) + N^2(\partial_t P^2 - \partial_t P^1)\Big). \tag{8.4}$$

Since $\partial_t P^2$ and N^2 are bounded on $[0,T] \times \mathbf{R}^3$ one finds that for $0 \le t \le T$

$$\partial_t f(t,x) + div\left(N_0(E^1 - E^2) \wedge (B^1 - B^2)\right) \le cf(t,x). \tag{8.5}$$

As in the existence part, the differential equation shows that $\partial_t(U^1 - U^2)$ is bounded with values in $L^2(\mathbf{R}^3)$. Let $Y_3(t) := \int_{\mathbf{R}^3} f(t,x)\,dx$. Intergate 8.5 over $([0,t] \times \mathbf{R}^3)$. Integration by parts is justified since $U^1 - U^2$ belongs to $H^1([0,t] \times \mathbf{R}^3)$. Since $U^1(0) = U^2(0)$ it follows that $Y_3(0) = 0$ and the above integration yields

$$Y_3(t) \le c \int_0^t Y_3(s)\,ds.$$

Gronwall's inequality implies that $Y_3(t) = 0$ for all $t \geq 0$ proving uniqueness. $\diamondsuit$

References

[BL] D.M. Bishop and B. Lam, *Ab initio* study of third order nonlinear optical properties of helium, Phys. Rev. A, 37, 464-469 (1988).

[BC] P. Butcher and D. Cotter, *The Elements of Nonlinear Optics*, Cambridge Univ. Press, Cambridge UK, 1990.

[CGT] R. Y. Chiao, E. Garmire, and C. H. Townes, Self-trapping of optical beams, Phys. Rev. Letters 13, 479-480 (1964).

[DLP] A. P. Dyshko, V. N. Lugovoi and A. M. Prokhorov, Development of an optical waveguide in the propagation of light in nonlinear medium Sov. Phys. J. Exp. Theor. Phys. 34, 1235-1241 (1972).

[DM] E. L. Dawes and J. H. Marburger, Computer studies in self-focusing, Phys. Rev. 179, 862-867 (1969).

[D] P. Donnat, Quelque contributions mathématiques en optique nonlinéaire, These, Ecole Polytechnique, Paris, 1994.

[DR] P. Donnat and J. Rauch, Dispersive nonlinear geometric optics, Jour. Math. Phys., to appear.

[FF] M.D. Feit and J.A. Fleck, Beam non paraxiality, filament formation and beam break up in the self-focusing of optical beam, J. Opt. Soc. Am. B 5, 633-640 (1988).

[G] R. T. Glassey, On the blow-up of solutions of the Cauchy problem for the nonlinear Schrödinger equation, J. Math. Physics 18, 1794-1707 (1977). [JMR] J.-L. Joly, G. Metivier, and J. Rauch, Global solvability of the anharmonic oscillator model from nonlinear optics, SIAM J. Math. Anal., 27, 905-913 (1996).

[H] L. Hormander, *Non-linear Hyperbolic Differential Equations*, Lecture Notes, Lund University, 1987.

[K] P. L. Kelley, Self-focussing of optical beams, Phys. Rev. Letters, 15,1005-1008 (1965).

[L] G.G. Luther, A.C. Newell, and J. V. Maloney, Normal dispersion arrests critical collapse, preprint.

[MP] V. Malkin and G. Papanicolaou, On self-focusing of short laser pulses, preprint (1993).

[M] J. H. Marburger, Self-focussing theory, Prog. Quant. Elect. 4, 35-110 (1975).

[NM] A. Newell and J. Moloney *Nonlinear Optics* Addison-Wesley, Reading Mass., 1992.

[PP] R. Pantell and H. Puthoff, *Fundamentals of Quantum Electronics*, Wiley N.Y., 1969

[Ra] J. Rauch, *Partial Differential Equations*, Springer-Verlag, N.Y., 1991.

[Re] M. Reed, *Abstract Non-Linear Wave Equations*, Springer-Verlag, N.Y., 1975.

[Rot1] J.E. Rothenberg, Pulse splitting during self-focussing in normally dispersive media, Opt. Letters, 17, 583-585 (1992).

[Rot2] J. E. Rothenberg, Breakdown of the slowly varying envelope approximation in the self-focussing of femtosecond pulses, Opt. Letters 17, 1340-1342 (1992).

[Ro] C. Rouyer, E. Mazataud, I. Allais, A. Pierre, S. Seznec, C. Sauteret, G. Mourou and A. Migus : generation of 50TW femtosecond pulses in a Ti:sapphire/Nd:glass chain, Opt. Letters 18, 214-216 (1993).

[Sh] Y.R. Shen, *T he Principles of Nonlinear Optics*, Wiley-Interscience, 1984.

[T] M. Taylor, *Pseudodifferential Operators and Nonlinear Optics*, Birkhäuser, Boston, 1991.

[Tal] V.I. Talanov, Self-focusing of wave beams in nonlinear media, Soviet Physics, J. Exper. Theor. Phys. **2**(1965) 138-141.

[Y] V. Yudovich, Nonstationary flows of an ideal incompressible fluid, *Zh. Vych. Math.* **3**(1963), 1032-1066.