A NOTE ON TWO DIMENSIONAL TRANSPORT WITH BOUNDED DIVERGENCE

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Abstract. We prove uniqueness for two dimensional transport across a noncharacteristic curve, under the hypothesis that the vector field is autonomous, bounded and with bounded divergence. We also obtain uniqueness for the Cauchy problem in $\mathbb{R}_t \times \mathbb{R}^2_x$ under an additional condition on the local direction of the vector field.

1. Introduction

In this note we study the uniqueness for the transport across a noncharacteristic curve in $\mathbb{R}^2_x$ and for the Cauchy problem relative to the transport equation in $\mathbb{R}_t \times \mathbb{R}^2_x$. Previous results by Bouchut and Desvillettes ([4]), Hauray ([11]) and Colombini and Lerner ([5] and [6]) show that uniqueness holds for the transport relative to an autonomous bounded divergence-free vector field and for the Cauchy problem in $\mathbb{R}_t \times \mathbb{R}^2_x$, under an additional condition on the local direction of the vector field. An extension to the non-divergence-free case is due to the first and the third authors (see [8]), who extended the theory to the case of autonomous bounded vector fields with bounded divergence and such that there exists a Lipschitz function θ, positive, bounded and bounded away from zero, such that

$$\mathrm{div}_x(\theta b) = 0 \,. \tag{1}$$

In [8] it is also conjectured that this hypothesis on the existence of a Lipschitz function θ could be removed.

We show that this conjecture has a positive answer: in fact, in this paper we show that we have uniqueness

- for the transport across a noncharacteristic curve in $\mathbb{R}^2_x$, with only the hypothesis that the vector field is autonomous, bounded and with bounded divergence;
- for the Cauchy problem in $\mathbb{R}_t \times \mathbb{R}^2_x$, under the hypothesis that the vector field is autonomous, bounded, with bounded divergence and satisfying (almost everywhere with respect to the one-dimensional Hausdorff measure $\mathscr{H}^1$ on $\mathbb{R}^2$) a local condition on its direction.

For the precise statements, see Theorems 3 and 5.

We remark that the two-dimensionality of these results cannot be dropped: the counter-examples given in [1], [7] and [9] show that there are autonomous divergence-free vector fields on $\mathbb{R}^3$ and nonautonomous divergence-free vector fields on $\mathbb{R}^2$ such that there is non-uniqueness in the Cauchy problem. In dimension $n \geq 3$ it is necessary to have some conditions on the derivatives of the vector field: the main references on this topic are the classical work by DiPerna and Lions [10] and the recent paper by Ambrosio [2].

We conclude this section with some remarks about the notation used. We denote by $\mathscr{L}^n$ the Lebesgue measure on $\mathbb{R}^n$ and by $\mathscr{H}^1$ the one-dimensional Hausdorff measure on $\mathbb{R}^n$. We denote by $\mathrm{div}_x b$ the spatial divergence (in the sense of distributions) of the vector field b. If b, u, and $\mathrm{div}_x b$ are locally bounded, then $b \cdot \nabla_x u$ denotes the distribution defined by

$$\langle b \cdot \nabla u, \varphi \rangle = -\langle bu, \nabla\varphi \rangle - \langle u \mathrm{div}_x b, \varphi \rangle \qquad \forall\, \varphi \in C_c^\infty(\mathbb{R}^2)\,. \tag{2}$$

2. Propagation across noncharacteristic surfaces in $\mathbb{R}^2_x$

We consider uniqueness in the Cauchy problem for the equation $b \cdot \nabla_x w = fw$ across a Lipschitz curve S in $\mathbb{R}^2$ oriented by an unit normal field ν, defined in a neighborhood of S. By this we mean that, for every $x_0 \in S$, there exists a Lipschitz function σ defined in a neighborhood U of x_0 such that $|\nabla\sigma| \geq \gamma > 0$ and $\nu = \nabla\sigma/|\nabla\sigma|$ $\mathscr{L}^2$-a.e. in U and $S = \{x \in U : \sigma(x) = 0\}$. We say that the vector field $b \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ is *positively transversal* to the surface S if, for every $x_0 \in S$, there exist a neighborhood U of x_0 in $\mathbb{R}^2$ and $\lambda > 0$ such that

$$b(x) \cdot \nu(x) \geq \lambda > 0 \qquad \text{for } \mathscr{L}^2\text{-a.e. } x \in U.$$

We will say that we have *uniqueness for the transport across the surface* S if, for every $x_0 \in S$, there exists a neighborhood V of x_0 in $\mathbb{R}^2$ such that, if we write V as the disjoint union

$$V = V^- \sqcup V^+ \sqcup (V \cap S)\,,$$

where V^+ (resp. V^-) is locally the half-space above (resp. under) the oriented S, and we suppose $\mathrm{div}_x b \in L^\infty(V)$ and $f \in L^\infty(V)$, then the only solution $w \in L^\infty(V)$ of the equation

$$b \cdot \nabla_x w = fw$$

with $w|_{V^-} \equiv 0$ is the function $w \equiv 0$.

We recall the following result, due to Colombini and Lerner ([5] and [6]), in the divergence-free case.

Theorem 1 (Colombini-Lerner)**.** *Let $c \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ with $\mathrm{div}_x c = 0$ and suppose that c is positively transversal to the curve S. Let $g \in L^\infty(\mathbb{R}^2)$. Then we have uniqueness for*

$$c(x) \cdot \nabla_x u(x) = g(x) u(x)$$

across the surface S.

The proof of this theorem is obtained observing that, since $\mathrm{div}_x c = 0$, the vector field is Hamiltonian, i.e. there exists a Lipschitz function $H : \mathbb{R}^2 \to \mathbb{R}$ such that $c = \nabla^\perp H$. This fact yields a change of coordinates which transforms the equation to $\partial w/\partial\rho = fw$. In the following theorem we generalize this result to the case with bounded divergence. The strategy is to apply the theorem of Colombini and Lerner to an auxiliary equation, whose vector field is divergence-free, and then to come back to the original problem. We first prove a preliminary lemma.

Lemma 2. *Suppose that* $b \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ *with* $\mathrm{div}_x b \in L^\infty(\mathbb{R}^2)$ *and suppose that* b *is positively transversal to the curve* S*. Then, for every* $y \in S$*, there exist an open neighbourhood* Λ *of* y *in* $\mathbb{R}^2$*, a positive function* $\rho \in L^\infty(\Lambda)$ *and a value* $\gamma > 0$ *such that* $\rho \geq \gamma$ *and* $\mathrm{div}_x(\rho b) = 0$ *on* Λ*.*

Proof. Approximate the vector field b in $L^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ with smooth vector fields b^ε, in such a way that every b^ε is positively transversal to the curve S in an open neighbourhood of y and $\|b^\varepsilon\|_\infty \leq C$. This can be done for example by convolving b with a standard approximate δ on $\mathbb{R}^2$. Let ρ^ε be a solution of the problem

$$\begin{cases} \mathrm{div}_x(\rho^\varepsilon b^\varepsilon) = 0 \\ \rho^\varepsilon = 1 \quad \text{ on } S \end{cases}$$

on an open neighbourhood of y. It is easy to show (for example writing the explicit solution of the regularized problems, via characteristics) that there is a common domain of definition Λ of all the ρ^ε and that $0 < \gamma \leq \rho^\varepsilon \leq 1/\gamma$ on Λ for every ε. Then $\{\rho^\varepsilon\}$ is weakly* compact in $L^\infty(\Lambda)$, so we can find a limit point ρ as $\varepsilon \to 0$ along some subsequence. Since $b^\varepsilon \to b$ strongly, we get $\rho^\varepsilon b^\varepsilon \stackrel{*}{\rightharpoonup} \rho b$ along the chosen subsequence. By linearity of the equation we have $\mathrm{div}_x(\rho b) = 0$ in the sense of distributions on Λ. Since ρ is a weak* limit point of $\{\rho^\varepsilon\}$, it satisfies the bounds $0 < \gamma \leq \rho \leq 1/\gamma$. □

Theorem 3. *Suppose that* $b \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$, $\mathrm{div}_x b \in L^\infty(\mathbb{R}^2)$*,* $f \in L^\infty(\mathbb{R}^2)$ *and* b *is positively transversal to the curve* S*. Let* $f \in L^\infty(\mathbb{R}^2)$*. Then we have uniqueness for*

$$b(x) \cdot \nabla_x w(x) = f(x)w(x) \tag{3}$$

across the surface S*.*

Proof. Write equation (3) in divergence form, recalling formula (2):

$$\mathrm{div}_x(wb) - w\mathrm{div}_x b = fw\,.$$

Since the statement is local, we can restrict the problem to the neighbourhood Λ of a point $y \in S$ given by Lemma 2 and fix a function ρ as in the lemma. Define $\widetilde{w} = w/\rho$ and substitute in the equation, obtaining

$$\mathrm{div}_x(\widetilde{w}(\rho b)) - \widetilde{w}\rho\mathrm{div}_x b = f\widetilde{w}\rho\,.$$

Since $\mathrm{div}_x(\rho b) = 0$ we get

$$(\rho b) \cdot \nabla_x \widetilde{w} = (\rho\mathrm{div}_x b + f\rho)\widetilde{w}\,.$$

The hypothesis of Theorem 1 are now satisfied since the vector field $c = \rho b$ is bounded, divergence-free and positively transversal to the curve S, because of the properties of the function ρ. The initial datum for $\widetilde{w}$ is zero, so we get that $\widetilde{w} = 0$ in a neighborhood of the surface S. Since $\rho \in L^\infty$ we have also $w = 0$ in a neighborhood of the surface S, that is the desired conclusion. □

Remark 4. In [8] it is essential that the preserved volume (the function θ in formula (1)) is *Lipschitz continuous*, because we need to give a (distributional) sense to the product of θ and $b \cdot \nabla_x u$ when u is just an L^∞ function. In our proof we do *not* multiply by the function ρ, but we modify *both the vector field and the solution*, obtaining a different equation for a different solution. In this way we avoid the Lipschitz regularity assumption on ρ.

3. The Cauchy problem in $\mathbb{R}_t \times \mathbb{R}^2_x$

In this section we study the Cauchy problem for the transport equation in $\mathbb{R}_t \times \mathbb{R}^2_x$

$$\begin{cases} \partial_t u(t,x) + b(x) \cdot \nabla_x u(t,x) = 0 \\ u(0,\cdot) = u_0\,. \end{cases} \tag{4}$$

Following Hauray [11], we consider the following condition (P_x) on the local direction of the vector field b in x:

$$(P_x) \qquad \begin{array}{c} \text{there exist } \xi \in \mathbb{S}^1,\ \alpha > 0 \text{ and } \varepsilon > 0 \text{ such that,} \\ \text{for } \mathscr{L}^2\text{-a.e. } y \in B_\varepsilon(x), \text{ we have } b(y) \cdot \xi \geq \alpha. \end{array}$$

In [11] it is proved that, if $b \in L^2_{\rm loc}(\Omega; \mathbb{R}^2)$, $\mathrm{div}_x b(x) = 0$ and (P_x) holds everywhere in Ω (or everywhere except a set of isolated points), then every bounded solution of the Cauchy problem (4) is *renormalized*, following the terminology introduced in [10]. This means that for every function $\beta : \mathbb{R} \to \mathbb{R}$ of class C^1 the following implication holds:

$$\partial_t u + b \cdot \nabla_x u = 0 \quad \Longrightarrow \quad \partial_t[\beta(u)] + b \cdot \nabla_x[\beta(u)] = 0\,.$$

It is a standard fact that the renormalization property implies uniqueness and stability for bounded solutions of the Cauchy problem (4) (for a complete treatment of the theory of renormalized solutions we refer to [10] and [2]).

We apply the same trick of the previous section to show that Hauray's proof extends to the case $b \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ with $\mathrm{div}_x b(x) \in L^\infty(\mathbb{R}^2)$, under the hypothesis that (P_x) holds, possibly except a closed $\mathscr{H}^1$-negligible set. As in the proof of Theorem 3, we will manipulate our equation until we reduce to a divergence-free case; then we show that renormalization holds for the modified equation, and from this we get uniqueness for the original problem.

Theorem 5. *Suppose that $b = (b_1, b_2) \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ with $\mathrm{div}_x b \in L^\infty(\mathbb{R}^2)$ and that $u_0 \in L^\infty(\mathbb{R}^2)$. Suppose that there exists an open set $\Omega \subset \mathbb{R}^2$ such that $\mathscr{H}^1(\mathbb{R}^2 \setminus \Omega) = 0$ and (P_x) holds for every $x \in \Omega$. Then we have uniqueness for the Cauchy problem* (4).

Proof. Fix $\bar{x} \in \Omega$ and consider ξ, α and ε given by $(P_{\bar{x}})$. After a rotation, we can suppose that $\xi = (1, 0)$. Up to a reduction of ε, we can suppose that there exists a positive function $\rho \in L^\infty(B_\varepsilon(\bar{x}))$, bounded away from 0, such that $\mathrm{div}_x(\rho b) = 0$ on $B_\varepsilon(\bar{x})$. This function can be constructed as in Lemma 2, taking for example as S the line orthogonal to the vector ξ.

Then

$$\begin{aligned} 0 &= \partial_t u + \mathrm{div}_x\big(bu\big) - u\mathrm{div}_x b \\ &= \partial_t u + \mathrm{div}_x\Big(\big(\rho b\big)\frac{u}{\rho}\Big) - u\mathrm{div}_x b \\ &= \partial_t\big(\widetilde{u}\rho\big) + \mathrm{div}_x\big(\widetilde{b}\widetilde{u}\big) - \widetilde{u}\rho\mathrm{div}_x b \qquad \text{in } \mathcal{D}'(\mathbb{R}_t \times B_\varepsilon(\bar{x})), \end{aligned}$$

where $\widetilde{b} = \rho b$ and $\widetilde{u} = u/\rho$. This yields

$$\rho(x)\partial_t\widetilde{u}(t,x) + \widetilde{b}(x)\cdot\nabla_x\widetilde{u}(t,x) = \widetilde{u}(t,x)\rho(x)\mathrm{div}_x b(x)\,. \tag{5}$$

Let $H \in \mathrm{Lip}(\mathbb{R}^2)$ be the Hamiltonian associated to $\widetilde{b}$, i.e. $\widetilde{b}_1 = -\partial H/\partial x_2$ and $\widetilde{b}_2 = \partial H/\partial x_1$. Thanks to the properties of ρ, it is immediate that $(P_{\bar{x}})$ holds also for $\widetilde{b}$, possibly with a smaller value α. Now consider $\varphi \in \mathrm{Lip}_c(\mathbb{R}\times B_\varepsilon(\bar{x}))$, where we denote by Lip_c the set of Lipschitz functions with compact support. We test our equation against φ to get

$$\begin{aligned} &\int_{\mathbb{R}\times B_\varepsilon(\bar{x})} \widetilde{u}(t,x)\Big[\rho(x)\partial_t\varphi(t,x) + \widetilde{b}(x)\cdot\nabla_x\varphi(t,x)\Big]\,dtdx \\ &\qquad + \int_{\mathbb{R}\times B_\varepsilon(\bar{x})} \widetilde{u}(t,x)\rho(x)\mathrm{div}_x b(x)\varphi(t,x)\,dtdx \;=\; 0\,. \end{aligned} \tag{6}$$

For $x = (x_1, x_2) \in B_\varepsilon(\bar{x})$ define

$$\Phi(x_1,x_2) := \big(x_1, H(x_1,x_2)\big)\,.$$

We are going to perform a change of variables in the formula (6), setting $(y_1,y_2) = \Phi(x_1,x_2) = (x_1, H(x_1,x_2))$. Observe that, since

$$D\Phi = \begin{pmatrix} 1 & 0 \\ \dfrac{\partial H}{\partial x_1} & \dfrac{\partial H}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \widetilde{b}_2 & -\widetilde{b}_1 \end{pmatrix}$$

and since we are supposing that $(P_{\bar{x}})$ holds with $\xi = (1,0)$, the function $\Phi : B_\varepsilon(\bar{x}) \to V = \Phi\big(B_\varepsilon(\bar{x})\big)$ is invertible (up to a further reduction of ε) and its inverse $\Psi = \Phi^{-1} : V \to B_\varepsilon(\bar{x})$ is Lipschitz. Let $\psi \in \mathrm{Lip}_c(\mathbb{R}\times V)$ and apply formula (6) with $\varphi(t,x) = \psi(t,\Phi(x))$. We can explicitly compute:

$$\begin{aligned} &\widetilde{b}(x)\cdot\nabla_x\varphi(t,x) \\ =\;& \widetilde{b}_1(\Psi(y))\big[\partial_{y_1}\psi(t,y) + \widetilde{b}_2(\Psi(y))\partial_{y_2}\psi(t,y)\big] + \widetilde{b}_2(\Psi(y))\Big[-\widetilde{b}_1(\Psi(y))\partial_{y_2}\psi(t,y)\Big] \\ =\;& \widetilde{b}_1(\Psi(y))\partial_{y_1}\psi(t,y)\,. \end{aligned}$$

Setting $v(t,y) = \widetilde{u}(t,\Psi(y))$ and computing $|\det(D\Psi)| = 1/\widetilde{b}_1(\Psi(y))$ we get

$$\int_{\mathbb{R}\times V} v(t,y)\Big[\rho(\Psi(y))\partial_t\psi(t,y) + \widetilde{b}_1(\Psi(y))\partial_{y_1}\psi(t,y)\Big]\frac{dtdy}{\widetilde{b}_1(\Psi(y))} + \int_{\mathbb{R}\times V} v(t,y)\rho(\Psi(y))\mathrm{div}_x b(\Psi(y))\psi(t,y)\frac{dtdy}{\widetilde{b}_1(\Psi(y))} = 0\,. \tag{7}$$

Define

$$R(y) := \frac{\rho(\Psi(y))}{\widetilde{b}_1(\Psi(y))} \qquad \text{and} \qquad Q(y) := \frac{\rho(\Psi(y))\mathrm{div}_x b(\Psi(y))}{\widetilde{b}_1(\Psi(y))}\,.$$

Then R and Q are positive, bounded and bounded away from zero on V. It follows that v satisfies the differential equation

$$R(y)\partial_t v(t,y) + \partial_{y_1} v(t,y) = Q(y)v(t,y)\,. \tag{8}$$

Then $v(t,y) = \widetilde{u}(t,\Psi(y))$ is a renormalized solution of (8) if and only if $\widetilde{u}(t,x)$ is a renormalized solution of (5), because we can follow all the above arguments backward. Since there is no differentiation with respect to y_2 in equation (8), it is equivalent to the fact that, for $\mathscr{L}^1$-a.e. $y_2 \in V_2$, we have

$$R(y_1,y_2)\partial_t v(t,y_1,y_2) + \partial_{y_1} v(t,y_1,y_2) = Q(y_1,y_2)v(t,y_1,y_2) \qquad \text{in } \mathcal{D}'(\mathbb{R}\times V_1), \tag{9}$$

if we suppose that V is the rectangle $V_1 \times V_2$, that is always possible up to a reduction of V. Now consider $w(t,z_1,z_2) = v(t,P^{-1}(z_1,z_2))$, where $\partial_{y_1}P(y_1,y_2) = R(y_1,y_2)$. Then equation (9) is equivalent to

$$\partial_t w(t,z_1,z_2) + \partial_{z_1} w(t,z_1,z_2) = \frac{Q(P^{-1}(z_1,z_2))}{R(P^{-1}(z_1,z_2))} w(t,z_1,z_2) \quad \text{in } \mathcal{D}'(\mathbb{R}\times\widetilde{V}_1), \text{ for a.e. } z_2 \in \widetilde{V}_2, \tag{10}$$

for every rectangle $\widetilde{V}_1 \times \widetilde{V}_2$ contained in $P(V_1 \times V_2)$. Now it is clear that every bounded solution of (10) is renormalized in a neighborhood of every point $(\bar{t},z_1) \in \mathbb{R}\times\widetilde{V}_1$.

This implies that the renormalization property holds for the equation (5) in the domain $\mathbb{R}_t \times \Omega$. Now recall that Lemma 4.1 of [6] (see also Proposition 3.4(i) of [3]) asserts that, under the assumptions of Theorem 5, if $U \in L^\infty(\mathbb{R}_t \times \mathbb{R}^2)$ satisfies (5) in $\mathbb{R}_t \times \Omega$ then it satisfies the same equation in the whole $\mathbb{R}_t \times \mathbb{R}^2$. Then we obtain that the renormalization property holds for the equation (5) in the whole $\mathbb{R}_t \times \mathbb{R}^2$. Hence we have uniqueness for the problem (5); this immediately implies uniqueness for the problem (4), since the two solutions u and $\widetilde{u}$ are related by the equality $\widetilde{u} = u/\rho$. □

Notice that our proof works also if we have a linear term of order zero in u on the right hand side of (4).

Remark 6. At the moment it is not clear to us if the problem (4) has uniqueness if we drop the assumption on the local direction of b. Non-uniqueness of the trajectories of b can occur, see the examples in Section 3 of [4].

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