Quadrature Estimates for Multidimensional Integrals *

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Abstract

We prove estimates for the error in the most straightforward discrete approximation to the integral of a compactly supported function of n variables. The methods use Fourier analysis and interpolation theory, and also make contact with classical lattice point estimates. We also prove error estimates for the approximation of the integral over an interval by the trapezoidal rule and the midpoint rule.

1 Introduction

Take $u \in L^\infty(\mathbb{T}^n)$, with $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$. We study the following basic problem in the theory of numerical integration: given further hypotheses on the behavior of u, how well do we approximate the mean value

$$Mu = (2\pi)^{-n} \int\limits_{\mathbb{T}^n} u(x)\, dx \tag{1.1}$$

by

$$\sigma_h u(x) = \nu^{-n} \sum_{\ell \in (\mathbb{Z}/\nu)^n} u(x + h\ell), \tag{1.2}$$

for an arbitrary $x \in \mathbb{T}^n$? Here we take $\nu \in \mathbb{Z}^+$ and $h = 2\pi/\nu$, and $\mathbb{Z}/\nu$ denotes the group of residue classes mod ν. Another way to write (1.2) is

$$\sigma_h u(x) = \nu^{-n} \sum_{\ell \in (\mathbb{Z}/\nu)^n} \tau_{h\ell} u(x), \qquad \tau_{h\ell} u(x) = u(x + h\ell). \tag{1.3}$$

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We estimate the error; in particular we estimate

$$\|\rho_h u\|_{L^\infty}, \qquad \rho_h u \;:=\; \sigma_h u - Mu. \tag{1.4}$$

While we have phrased the basic problem as one of approximating an integral of a function over $\mathbb{T}^n$, a more common formulation is to approximate the integral of a compactly supported function u on $\mathbb{R}^n$. This is transformed to the problem described above by the device of scaling so that $\operatorname{supp} u \subset [-\pi,\pi]^n$ and identifying the opposite sides of this n-dimensional cube, so u is identified with a function on $\mathbb{T}^n$. (In case $n=1$, an alternative reflection method, discussed in §6, has advantages.)

The problem of estimating the remainder $\rho_h u$ defined in (1.4) is classical in numerical analysis, and of continued interest. We mention the articles [2] and [3], which in turn have further references. Useful classical bounds for the error are

$$\|\rho_h u\|_{L^\infty} \;\le\; \begin{cases} Ch^r\|u\|_{C^r}, & r\in\mathbb{N}, \\ Ch^r\|u\|_{H^{r,1}}, & r\in\mathbb{N},\ r>n \text{ (or } n=1) \end{cases}. \tag{1.5}$$

for u supported in $(-\pi,\pi)^n$. For $n=1$ this error bound is often not far from the correct order of convergence.

To see some shortcomings, which motivate a more thorough investigation, begin with the observation that the space $H^{r,1}$ is invariant under diffeomorphisms of $\mathbb{R}^n$ that are linear outside some compact set. However, the rate of convergence of the quadrature error to zero does not have this invariance. Consider for example the case of $u=\chi_\Omega$, the characteristic function of an open set. The quadrature (1.2) counts the number of lattice points from $h\mathbb{Z}^n$ that belong to Ω. For a domain whose boundary contains an open piece of hyperplane whose normal has components with rational ratio, one has no better estimate than $\|\rho_h u\|_{L^\infty}\le Ch$. However, there are domains of such a type that are diffeomorphic to the unit ball, for which the error converges to zero as a strictly larger power of h. In a nontrivial way, the affine geometry of $\mathbb{R}^n$ plays an important role in the behavior of the error, and one of the things we achieve is to capture this dependence in our estimates. The key aspect of our attack is to make strong use of Fourier series, which by its definition depends on the affine structure. A key element of many of our estimates is the pointwise decay of the Fourier coefficients. The pointwise decay of the Fourier coefficients for the characteristic function of a ball is more rapid than that for χ_Ω when the boundary $\partial\Omega$ has flat pieces. In contrast, both characteristic functions belong to BV and no better, so their Sobolev regularity is the same.

Start with the derivation of a formula for the error $\rho_h u$ in terms of Fourier series. The Fourier coefficients of a function u on $\mathbb{T}^n$ are given by

$$\hat{u}(j) = (2\pi)^{-n} \int\limits_{\mathbb{T}^n} u(x) e^{-ij\cdot x}\, dx, \quad j \in \mathbb{Z}^n; \tag{1.6}$$

in particular $Mu = \hat{u}(0)$. From (1.2) it follows easily that

$$\widehat{\sigma_h u}(j) = \pi_\nu(j)\hat{u}(j), \quad \pi_\nu(j) = 1 \ \text{ if } \nu | j, \ 0 \text{ otherwise.} \tag{1.7}$$

Hence

$$\rho_h u(x) = \sum_{j\neq 0} \hat{u}(\nu j) e^{i\nu j\cdot x}, \qquad \nu = \frac{2\pi}{h}. \tag{1.8}$$

This basic identity will be exploited in several ways in subsequent sections. We will also use for the following identity. Suppose that $\Psi \in C_0^\infty(\mathbb{R}^n)$ satisfies $\Psi(\xi) = 0$ for $|\xi| \geq 1$, and $\Psi(\xi) = 1$ for $|\xi| \leq 1/2$. Denote by $\Psi(hD)$ the operator on f that takes $\hat{f}(j)$ to $\Psi(hj)\hat{f}(j)$. Then,

$$\begin{aligned} \rho_h u &= \big(I - \Psi(hD)\big)\sigma_h u \\ &= \nu^{-n} \sum_{\ell \in (\mathbb{Z}/\nu)^n} \tau_{h\ell}\big(I - \Psi(hD)\big)u. \end{aligned} \tag{1.9}$$

The paper is organized as follows. In §2 we estimate $\|\rho_h u\|_{L^\infty}$ in terms of various function space estimates on u. We show that (1.9) yields estimates for Hölder continuous functions, and (1.8) yields estimates for functions whose Fourier coefficients have certain decay. The latter class contains functions whose sufficiently many distributional derivatives are integrable, but as we show there are advantages in the extra generality of the results as they are presented in Proposition 2.1. They allow us to apply complex interpolation techniques, and obtain estimates for functions in Besov spaces, which in the low regularity context are much sharper than the estimates in terms of Hölder norms.

In §3 we employ further harmonic analysis techniques to estimate $\|\rho_h u\|_{L^\infty}$ for a class of piecewise regular functions on $\mathbb{T}^n$ with a jump across a smooth surface. In addition to the estimates of §2, we use techniques related to lattice point estimates, producing a unified analysis of lattice point and quadrature problems.

In §4 we estimate $\|\rho_h u\|_{L^p}$, particularly for $p = 2$, and deduce results on the size of the error when Mu is approximated by $\sigma_h u(x)$ evaluated at a random point $x \in \mathbb{T}^n$, at least with high probability if not with certainty. Comparisons are made with the expected error in the Monte Carlo method.

In §5 we strengthen estimates $\|\rho_h u\|_{L^\infty} \le C\,h^r$ to $\|\rho_h u\|_{L^\infty} = o(h^r)$, under appropriate hypotheses on u.

Section 6 is the only section of this paper in which we focus on the one-dimensional case. We return to the setting of §2, specialize to dimension $n = 1$, and derive analogues of the estimates of §2 for the approximation of the integral over an interval by the trapezoidal rule and the midpoint rule.

2 L^∞ error estimates

We can use the results (1.8)–(1.9) to establish some estimates on $\|\rho_h u\|_{L^\infty}$, using the following function spaces. For $r > 0$ define F^r by,

$$u \in F^r \Longleftrightarrow |\hat{u}(j)| \le C(1+|j|)^{-r}, \qquad \|u\|_{F^r} := \sup_j\,(1+|j|)^r |\hat{u}(j)|. \tag{2.1}$$

Denote by C^r_* the scale of Zygmund spaces of functions on $\mathbb{T}^n$. If r is not an integer, $C^r_* = C^r$, defined as the space of functions whose kth derivatives are Hölder continuous of exponent s if $r = k + s$, $k \in \mathbb{Z}^+$, $s \in (0,1)$. In (2.2)–(2.3) and subsequent estimates, "C" will denote constants that differ from line to line. Such constants C will be independent of h and u, but not independent of r.

Proposition 2.1 *We have*

$$\|\rho_h u\|_{L^\infty} \le Ch^r\,\|u\|_{C^r_*}, \quad r > 0, \tag{2.2}$$

and

$$\|\rho_h u\|_{L^\infty} \le Ch^r\,\|u\|_{F^r}, \quad r > n. \tag{2.3}$$

Proof. From (1.8) we have

$$\|\rho_h u\|_{L^\infty} \le \big\|\big(I - \Psi(hD)\big)u\big\|_{L^\infty}, \tag{2.4}$$

which readily yields (2.2). Meanwhile (1.7) implies

$$\|\rho_h u\|_{L^\infty} \le \sum_{j\ne 0} |\hat{u}(\nu j)| \le \sum_{j\ne 0} (1+|\nu j|)^{-r}\,\|u\|_{F^r}, \tag{2.5}$$

which gives (2.3). □

We mention connections with previous work. For $n = 1$, $r \in (0,1)$, the estimate (2.2) is given in [2]. Actually [2] works on an interval, which makes no difference in this context; see §6 for further discussion of this point. Estimates similar to (2.3) are given in [8] for $n = 1$ and also in [2], for $r = 2$, and in [10] (Lemma 6.2) for $n = 2$, except that in place of F^r these authors use L^1-Sobolev spaces. The F^r-norm is weaker than the corresponding Sobolev space norm, is not diffeomorphism invariant, and will have useful consequences for our subsequent results. We also mention the work of [9], which uses (1.8), in the case $n = 1$; this work analyzes the case where u is C^∞ except for isolated singularities, of algebraic or algebraic/logarithmic type and produces an asymptotic expansion for $\rho_h u(0)$ in such cases. We discuss this later in this section.

In addition, there is work for lattices in $\mathbb{T}^n$ other than $(\mathbb{Z}/\nu)^n$. See [6], §5.5 and [7], §6.5, and references given there, in particular [11]. Variants of (1.8) also play a role in these works.

For $r > n$, (2.3) is stronger than (2.2). For $r < n$ one gets no information from (2.3); the result (2.2) applies for $r \in (0, n)$, but this result is rather crude. The next result gives an improvement of (2.2), making use of the Besov spaces $B^r_{p,\infty}$ characterized by $u \in B^r_{p,\infty}$ if and only if,

$$\|\psi_k(D)u\|_{L^p} \le C\, 2^{-kr}, \tag{2.6}$$

where $\{\psi_k(\xi) : k \ge 0\}$ is a Littlewood-Paley partition of unity, with $\operatorname{supp} \psi_k \subset \{\xi \in \mathbb{R}^n : 2^{k-1} \le |\xi| \le 2^{k+1}\}$ for $k \ge 1$, $\operatorname{supp} \psi_0 \subset \{\xi : |\xi| \le 2\}$.

Proposition 2.2 *For $p \in [1, \infty)$,*

$$\|\rho_h u\|_{L^\infty} \le Ch^r \|u\|_{B^r_{p,\infty}}, \quad \textit{provided} \quad rp > n. \tag{2.7}$$

Proof. Note that $C^r_* = B^r_{\infty,\infty}$ and $B^r_{1,\infty} \subset F^r$, so (2.2)–(2.3) yield

$$\begin{aligned} \|\rho_h u\|_{L^\infty} &\le Ch^{r_0} \|u\|_{B^{r_0}_{\infty,\infty}}, \quad r_0 > 0, \\ \|\rho_h u\|_{L^\infty} &\le Ch^{r_1} \|u\|_{B^{r_1}_{1,\infty}}, \quad r_1 > n. \end{aligned} \tag{2.8}$$

The estimate (2.7) would follow immediately from (2.8) if we had

$$[B^{r_0}_{\infty,\infty}, B^{r_1}_{1,\infty}]_\theta = B^r_{p,\infty}, \quad r = (1-\theta)r_0 + \theta r_1, \quad \frac{1}{p} = \theta, \quad 0 < \theta < 1. \tag{2.9}$$

Here, the left side is the complex interpolation space of A. Calderon. However, (2.9) is not quite true (cf. [14], pp. 67–73), so further argument is required. Define

$$u_k = \psi_k(D)u, \qquad v_k(z) = \frac{u_k}{|u_k|} |u_k|^{zp}, \tag{2.10}$$

for $z \in \Omega = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ (setting $v_k(z)(x) = 0$ where $u_k(x) = 0$), and then form

$$w_k(z) = \tilde{\psi}_k(D) v_k(z), \tag{2.11}$$

where $\tilde{\psi}_k(\xi) = \sum_{|k-\ell|\le 3} \psi_\ell(\xi)$. Given $\alpha \in \mathbb{R}$ (to be chosen shortly), set

$$w(z) = \sum_{k\ge 0} 2^{-k\alpha(z-\theta)} w_k(z). \tag{2.12}$$

We see that

$$w(\theta) = u, \quad \text{if } \theta = \frac{1}{p}. \tag{2.13}$$

Note that for $y \in \mathbb{R}$,

$$\begin{aligned} \|2^{-k\alpha(iy-\theta)} w_k(iy)\|_{L^\infty} &\le C\, 2^{-k(r-\alpha\theta)}, \\ \|2^{-k\alpha(1+iy-\theta)} w_k(1+iy)\|_{L^1} &\le C\, 2^{-k(r+\alpha(1-\theta))}. \end{aligned} \tag{2.14}$$

Hence if we take $\alpha = r_1 - r_0$ in (2.12), we have

$$\|w(iy)\|_{B^{r_0}_{\infty,\infty}},\ \|w(1+iy)\|_{B^{r_1}_{1,\infty}} \ \le\ C\|u\|_{B^r_{p,\infty}}, \tag{2.15}$$

with r_j, r, p, and θ related as in (2.9). One also sees that, for $z \in \Omega$, $w(z)$ is bounded in $B^{r_0}_{\infty,\infty} + B^{r_1}_{1,\infty}$. One does not have w continuous on $\overline{\Omega}$ with values in $B^{r_0}_{\infty,\infty} + B^{r_1}_{1,\infty}$, which is why this argument does not prove the inclusion of $B^r_{p,\infty}$ in the left side of (2.9). But, one does have continuity with values in $B^{r_0-\varepsilon}_{\infty,\infty} + B^{r_1-\varepsilon}_{1,\infty}$ for each $\varepsilon > 0$, and hence continuity with values in $C(\mathbb{T}^n)$, given that $r_0 > 0$ and $r_1 > n$.

The upshot is that for each $x \in \mathbb{T}^n$, $\rho_h w(z)(x)$ is a bounded, continuous function on $\overline{\Omega}$, holomorphic on Ω, satisfying

$$\begin{aligned} |\rho_h w(iy)(x)| &\ \le\ C h^{r_0} \|u\|_{B^r_{p,\infty}}, \\ |\rho_h w(1+iy)(x)| &\ \le\ C h^{r_1} \|u\|_{B^r_{p,\infty}}, \end{aligned} \tag{2.16}$$

by (2.8) and (2.15), whenever r, r_j, and p are related as in (2.9), and as long as $r_0 > 0,\ r_1 > n$. The estimate

$$|\rho_h u(x)| \ \le\ C h^r \|u\|_{B^r_{p,\infty}} \tag{2.17}$$

then follows from the three-lines lemma. Note that

$$rp = r_1 + \frac{1-\theta}{\theta} r_0, \tag{2.18}$$

so given $r > 0,\ p \ge 1$, there exist $r_0 > 0,\ r_1 > n$ such that this argument applies precisely when $rp > n$. □

Since the L^p-Sobolev space $H^{r,p}(\mathbb{T}^n)$ is contained in $B^r_{p,\infty}(\mathbb{T}^n)$, we have:

Corollary 2.3 *For* $p \in (1, \infty)$,

$$\|\rho_h u\|_{L^\infty} \le C h^r \|u\|_{H^{r,p}}, \quad \textit{provided} \;\; rp > n. \tag{2.19}$$

The estimate (2.19) is mainly interesting for $r \in (n/p, n]$, since for $r > n$ the estimate (2.3) is stronger. See (5.6) for a further strengthening of (2.19).

We mention that for $r = n = 1$, (2.19) can be strengthened to

$$\|\rho_h u\|_{L^\infty} \le C h \|u\|_{BV}, \tag{2.20}$$

given u continuous and of bounded variation on $\mathbb{T}^1$. See [2], Theorem 1.6, which is given an elementary proof. (Again, this result works on an interval.) We also recall the well known elementary result

$$\|\rho_h u\|_{L^\infty} \le C h^2 \|u'\|_{BV}, \tag{2.21}$$

given that u is Lipschitz and u' has bounded variation on $\mathbb{T}^1$ (cf. [2], [13]). The case $n = 1, \; r = 2$ of (2.3) is slightly stronger than (2.21).

We return to the error estimate (2.3) and discuss further its advantages over the analogous estimates involving L^p-Sobolev regularity, i.e.,

$$\|\rho_h u\|_{L^\infty} \le C h^r \|u\|_{H^{r,p}}, \quad r > n, \tag{2.22}$$

valid for all $p \in [1, \infty)$ if $r \in \mathbb{N}$ (so one might as well take $p = 1$), and valid for all $p \in (1, \infty)$ if $r \notin \mathbb{N}$. If $r \notin \mathbb{N}$, one could also use

$$\|\rho_h u\|_{L^\infty} \le C h^r \|u\|_{\mathfrak{h}^{r,1}}, \tag{2.23}$$

where $\mathfrak{h}^{r,1}$ is the Hardy-Sobolev space. For $n = 1$, one could define $H^{r,1}$ to be $(1 + \partial)^{-r} L^1(\mathbb{T}^1)$, where $(1+\partial)^{-r} \sum a_k e^{ikx} = \sum (1 + ik)^{-r} a_k e^{ikx}$, and use $H^{r,1}$ in place of $\mathfrak{h}^{r,1}$ in (2.23), but for $n > 1$ the space $H^{r,1}$ is not well defined when r is not an integer. One has clearly

$$H^{r,p} \subset F^r, \quad \mathfrak{h}^{r,1} \subset F^r, \tag{2.24}$$

so (2.22) and (2.23) follow from (2.3). We give some examples to illustrate how (2.3) can yield a significantly larger exponent on h than either (2.22) or (2.23).

To begin, take $n > 1$ and let $\Omega \subset \mathbb{T}^n$ be a smoothly bounded region whose boundary $\partial\Omega$ has nowhere vanishing Gauss curvature. Then a standard stationary phase calculation yields for the characteristic function χ_Ω,

$$\chi_\Omega \;\in\; F^{(n+1)/2}, \tag{2.25}$$

while $\nabla\chi_\Omega$ is a measure supported on $\partial\Omega$, so χ_Ω is not quite in $H^{1,1}(\mathbb{T}^n)$. More generally, for $a \in \mathbb{R}^+$, not an even integer, with $\rho(x) = \operatorname{dist}(x, \mathbb{T}^n \setminus \Omega)$, we can take a function

$$u_a \in C^\infty(\mathbb{T}^n \setminus \partial\Omega), \qquad u_a(x) = \rho(x)^a \ \text{ near } \ \partial\Omega, \tag{2.26}$$

and we have

$$u_a \in F^{a+(n+1)/2}, \quad u_a \in \mathfrak{h}^{a+1-\varepsilon,1}, \ \ \forall\, \varepsilon > 0, \quad u_a \notin \mathfrak{h}^{a+1,1}. \tag{2.27}$$

As long as $a > (n-1)/2$, (2.3) yields a much stronger estimate than (2.23), namely

$$\|\rho_h u_a\|_{L^\infty} \le C h^{a+(n+1)/2}, \qquad a > \frac{n-1}{2}. \tag{2.28}$$

For $a < (n-1)/2$, neither (2.3) nor (2.22)–(2.23) are applicable. We do have

$$\chi_\Omega \in B^{1/p}_{p,\infty}, \qquad u_a \in B^{a+1/p}_{p,\infty}, \tag{2.29}$$

so (2.7) applies as long as $pa + 1 > n$, yielding

$$\|\rho_h u_a\|_{L^\infty} \le C_p h^{a+1/p}, \quad \forall\, p > \frac{n-1}{a}, \tag{2.30}$$

i.e.,

$$\|\rho_h u_a\|_{L^\infty} \le C_\varepsilon h^{an/(n-1)-\varepsilon}, \quad \forall\, \varepsilon > 0. \tag{2.31}$$

The estimate (2.31) holds regardless of whether $\partial\Omega$ satisfies the Gauss curvature condition assumed above. Under this curvature hypothesis, one can do better with the following interpolation argument. For $z \in \mathbb{C}$, set

$$v_{a,z} = (1-\Delta)^{z/2} u_a. \tag{2.32}$$

Assume $a \in (0, (n-1)/2)$. Then, using (2.29) we get for some $A < \infty$

$$\|v_{a,a-\varepsilon+it}\|_{L^\infty(\mathbb{T}^n)} \le C_\varepsilon e^{A|t|}, \quad \forall\, \varepsilon > 0, \tag{2.33}$$

while also, via (2.27),

$$\|v_{a,a-(n-1)/2-\varepsilon+it}\|_{F^{n+\varepsilon}} \le C_\varepsilon, \quad \forall\, \varepsilon > 0. \tag{2.34}$$

From (2.33)–(2.34) we get

$$\begin{aligned} \|\rho_h v_{a,a-\varepsilon+it}\|_{L^\infty} &\le C_\varepsilon e^{A|t|}, \\ \|\rho_h v_{a,a-(n-1)/2-\varepsilon+it}\|_{L^\infty} &\le C_\varepsilon h^{n+\varepsilon}, \end{aligned} \tag{2.35}$$

and the three lines lemma yields an estimate for $\|\rho_h u_a\|_{L^\infty} = \|\rho_h v_{a,0}\|_{L^\infty}$, namely

$$\|\rho_h u_a\|_{L^\infty} \le C_\varepsilon h^{2an/(n-1)-\varepsilon}, \quad \forall\, \varepsilon > 0. \tag{2.36}$$

Note the advantage over (2.31). Clearly (2.36) is at least close to sharp for a close to $(n-1)/2$. However, for a close to 0, methods developed in §3 yield significantly stronger estimates.

The examples just considered, involving u_a in (2.26), have focused on the case $n > 1$. Now we look at the case $n = 1$. In this case, we can take Ω to be an interval in $\mathbb{T}^1$ and consider χ_Ω and u_a as in (2.25)–(2.26), and we still have (2.27)–(2.28), i.e.,

$$u_a \in F^{a+1}, \quad u_a \in \mathfrak{h}^{a+1-\varepsilon,1},\ \forall\, \varepsilon > 0, \tag{2.37}$$

and, by (2.3),

$$\|\rho_h u_a\|_{L^\infty} \le C h^{a+1}, \quad a > 0. \tag{2.38}$$

Note that an application of (2.23) to (2.37) yields $\|\rho_h u_a\|_{L^\infty} \le C_\varepsilon h^{a+1-\varepsilon}$, $\forall\, \varepsilon > 0$, which is only slightly weaker than (2.38). The following result shows that there are functions u on $\mathbb{T}^1$ for which (2.3) provides better estimates on $\|\rho_h u\|_{L^\infty}$ than does (2.23), by a factor $h^{1/2}$.

Proposition 2.4 *There exist functions u on $\mathbb{T}^1$ with the properties*

$$u \in F^{3/2}, \quad \textit{but} \ \ u \notin H^{1,1}. \tag{2.39}$$

Proof. We use the following result, whose proof is in [5].

Lemma 2.5 *Let $\mathcal{A} = \prod_{k=-\infty}^{\infty} S^1$, with its product probability measure, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Given $(c_k) \in \ell^\infty$, $(c_k) \notin \ell^2$, form for each $\alpha = (\alpha_k) \in \mathcal{A}$,*

$$F_\alpha = \sum_{k=-\infty}^{\infty} \alpha_k c_k e^{ikx}, \quad F_\alpha \in H^{-1/2-\varepsilon}(\mathbb{T}^1),\ \forall\, \varepsilon > 0. \tag{2.40}$$

Then

$$F_\alpha \notin L^1(\mathbb{T}^1), \quad \textit{for a.e. } \alpha \in \mathcal{A}. \tag{2.41}$$

We proceed to prove Proposition 2.4. Consider

$$F_\alpha \ := \sum_{k\neq 0} \frac{\alpha_k}{k^{1/2}} e^{ikx}. \tag{2.42}$$

Lemma 2.5 applies, so (2.41) holds. Given an α so that $F_\alpha \notin L^1(\mathbb{T}^1)$, take

$$u \ := \ \sum_{k\neq 0} \frac{\alpha_k}{k^{3/2}} e^{ikx}, \quad \text{so} \ \ F_\alpha \ := \ \frac{1}{i}\partial_x u. \tag{2.43}$$

Then we have (2.39), and Proposition 2.4 is proven. □

Applying Fourier multiplication by powers of k, we readily obtain from (2.39) functions u_r on $\mathbb{T}^1$ satisfying

$$u_r \in F^r \quad \text{but} \ \ u_r \notin \mathfrak{h}^{r-1/2,1}. \tag{2.44}$$

Now return to the discussion of u_a, given by (2.26), in case $n = 1$ and $\Omega \subset \mathbb{T}^1$ is an interval, in which case we have (2.37)–(2.38). The result (2.38) was obtained by [9] in this context, making use of the identity (1.5) (in case $n = 1$). In fact, [9] pursued the analysis of $\rho_h u(0)$ for functions of one variable with a discrete set of simple singularities, of such type as $(x - x_j)^a$, perhaps with a factor of $\log(x - x_j)$ thrown in. One of the major points made in [9] is that for such functions u, $\hat{u}(k)$ has a complete asymptotic expansion, which enables one to get a complete asymptotic expansion for $\rho_h u(0)$ as $h \to 0$. This expansion is truncated for more general functions such as $u_a f$, with f of limited regularity; as noted in [1] (p. 255), one has

$$\|\rho_h(u_a f)\|_{L^\infty} \le C h^{1+a}, \quad \text{for} \ \ a \in (0,1), \ f \in C^2(\mathbb{T}^1). \tag{2.45}$$

Actually, [9] and [1] work with the trapezoidal rule on functions on an interval, say $[0,\pi]$, though the reflection argument discussed in §5 of this paper reduces this result to its analogue (2.45) on $\mathbb{T}^1$, at least if $C^2(\mathbb{T}^1)$ is enlarged to $C^{1,1}(\mathbb{T}^1)$. In fact, we claim that the following more general result holds:

$$\|\rho_h(u_a f)\|_{L^\infty} \le C h^{1+a}, \quad \text{for} \ \ a \in (0,1), \ f \in F^{1+a}(\mathbb{T}^1). \tag{2.46}$$

This is a consequence of the following result, which we formulate in the n-dimensional context.

Proposition 2.6 *Given $r > n$, we have*

$$u, v \in F^r(\mathbb{T}^n) \quad \Longrightarrow \quad uv \in F^r. \tag{2.47}$$

Proof. We need to estimate

$$\hat{w}(k) = \sum_{\ell \in \mathbb{Z}^n} \hat{u}(\ell)\hat{v}(k-\ell). \tag{2.48}$$

To get this, let $A_k = \{\ell \in \mathbb{Z}^n : (\ell - k/2) \cdot k \le 0\}$ and $B_k = \{\ell \in \mathbb{Z}^n : (\ell - k/2) \cdot k > 0\}$, and estimate the two pieces of

$$\hat{w}(k) = \sum_{A_k} \hat{u}(\ell)\hat{v}(k-\ell) + \sum_{B_k} \hat{u}(\ell)\hat{v}(k-\ell). \tag{2.49}$$

Setting $\langle k \rangle = (1+|k|^2)^{1/2}$, we have

$$\begin{aligned} \Bigl| \sum_{A_k} \hat{u}(\ell)\hat{v}(k-\ell) \Bigr| &\le C\langle k\rangle^{-r} \sum |\hat{u}(\ell)| \\ &\le C\langle k\rangle^{-r}, \end{aligned} \tag{2.50}$$

provided $r > n$, which implies $\sum |\hat{u}(\ell)| < \infty$. A similar estimate works for the last term on the right side of (2.49), giving

$$|\hat{w}(k)| \ \le \ C\langle k\rangle^{-r}, \tag{2.51}$$

yielding (2.47). □

In view of results to be presented in §3, it is of interest to have the following extension of Proposition 2.6. The proof is a straightforward variant of the preceding argument.

Proposition 2.7 *Given $r > n$ and $s \in [0, r]$, we have*

$$u \in F^r(\mathbb{T}^n),\ v \in F^s(\mathbb{T}^n) \quad \Longrightarrow \quad uv \in F^s. \tag{2.52}$$

We end this section with another perspective on how to use the identity (1.7) to estimate the error $\rho_h u$. Namely, set

$$G_r(x) \ := \ \frac{1}{(2\pi)^n} \sum_{j\neq 0} \frac{e^{-ij\cdot x}}{|j|^r}, \qquad G_r^\nu(x) \ := \ G_r(\nu x). \tag{2.53}$$

Then (1.7) is equivalent to

$$\rho_h u \ = \ \Bigl(\frac{h}{2\pi}\Bigr)^r (\Lambda^r u) * G_r^\nu, \tag{2.54}$$

(with $h = 2\pi/\nu$), where Λ^r is defined by

$$(\Lambda^r u)\hat{}\,(j) = |j|^r \hat{u}(j). \tag{2.55}$$

This yields the following result.

Proposition 2.8 *Fix $r \in (0,\infty)$ and let X^r be a Banach space of functions on $\mathbb{T}^n$ (with translation-invariant norm) with the property that*

$$\{G_r^\nu : \nu \in \mathbb{N}\} \ \text{ is bounded in } \ X^r, \text{ with norm } \le K. \tag{2.56}$$

Let $(X^r)'$ denote the dual of X^r. Then

$$\Lambda^r u \in (X^r)' \quad \Longrightarrow \quad \|\rho_h u\|_{L^\infty} \ \le \ K\Bigl(\frac{h}{2\pi}\Bigr)^r \|\Lambda^r u\|_{(X^r)'}. \tag{2.57}$$

To illustrate the application of Proposition 2.8, let us note that

$$\begin{aligned} r > n \quad &\Longrightarrow \quad G_r \in C(\mathbb{T}^n) \\ &\Longrightarrow \quad G_r^\nu \ \text{ bounded in } \ C(\mathbb{T}^n), \end{aligned} \tag{2.58}$$

and the dual of $C(\mathbb{T}^n)$ is $\mathcal{M}(\mathbb{T}^n)$, the space of finite Borel measures on $\mathbb{T}^n$. Hence Proposition 2.8 implies

$$r > n, \ \Lambda^r u \in \mathcal{M}(\mathbb{T}^n) \quad \Longrightarrow \quad \|\rho_h u\|_{L^\infty} \le C h^r \|\Lambda^r u\|_{\mathcal{M}(\mathbb{T}^n)}. \tag{2.59}$$

This result is also a consequence of (2.3).

In this setting, one can cast the task of determining when $\|\rho_h u\|_{L^\infty} \le C h^r$ as that of determining Banach spaces X^r satisfying (2.56), together with the task of determining their duals.

3 Further L^∞ error estimates

We obtain estimates on $\|\rho_h u\|_{L^\infty}$ when we are given that $u \in F^r$ for some $r \in (0,n)$, and we are also given some further information on u. We use the following strategy. Let J_ε denote a Friedrichs mollifier. That is, pick $\varphi \in \mathcal{S}(\mathbb{R}^n)$, with $\varphi(0) = 1$, and set $J_\varepsilon = \varphi(\varepsilon D)$. We will estimate $\|\rho_h J_\varepsilon u\|_{L^\infty}$ in terms of $\|u\|_{F^r}$. Then we will estimate $\|\rho_h(u - J_\varepsilon u)\|_{L^\infty}$ by another method, depending on the nature of the extra information on u. We take $\varepsilon = h^{1+\mu}$, for some $\mu > 0$, and find an optimal value of μ, where the two sorts of estimates have the same order of magnitude. This is a natural generalization of a strategy used for lattice point estimates, where u is the characteristic function of a region, with a “nice” boundary.

Our estimate begins with

$$\rho_h J_\varepsilon u \ = \ \sum_{j \ne 0} \hat{u}(\nu j)\, \varphi(\nu \varepsilon j)\, e^{i\nu j \cdot x}. \tag{3.1}$$

Therefore, for each positive integer N,

$$\begin{aligned}\|\rho_h J_\varepsilon u\|_{L^\infty} &\le C_N \sum_{j\neq 0} |\hat{u}(\nu j)|\,(1+|\nu\varepsilon j|)^{-N} \\ &\le C_N \|u\|_{F^r} \sum_{j\neq 0} |\nu j|^{-r}(1+|\nu\varepsilon j|)^{-N} \\ &\approx C_N \|u\|_{F^r} \int\limits_{|\xi|\ge 1} |\nu\xi|^{-r}(1+|\nu\varepsilon\xi|)^{-N}\,d\xi \\ &= C_N \|u\|_{F^r}\varepsilon^r(\nu\varepsilon)^{-n} \int\limits_{|\zeta|\ge\nu\varepsilon} |\zeta|^{-r}(1+|\zeta|)^{-N}\,d\zeta,\end{aligned} \tag{3.2}$$

using $\zeta = \nu\varepsilon\xi$. If $r \in (0,n)$, take $\varepsilon = h^{1+\mu}$ to find,

$$\begin{aligned}\|\rho_h J_\varepsilon u\|_{L^\infty} &\le C\varepsilon^r(\nu\varepsilon)^{-n}\|u\|_{F^r} \\ &= C' h^{r-(n-r)\mu}\|u\|_{F^r}.\end{aligned} \tag{3.3}$$

Here is a first application of the estimate (3.3).

Proposition 3.1 *Assume*

$$u \in F^r \cap C^s_*, \qquad 0 < s < r < n. \tag{3.4}$$

Then

$$\|\rho_h u\|_{L^\infty} \le C h^\gamma(\|u\|_{F^r} + \|u\|_{C^s_*}), \quad \gamma = s\,\frac{n+2(r-s)}{n+(r-s)}. \tag{3.5}$$

Proof. In addition to (3.3), we have

$$\|\rho_h(u - J_\varepsilon u)\|_{L^\infty} \le \|u - J_\varepsilon u\|_{L^\infty} \le C\varepsilon^s\|u\|_{C^s_*}, \tag{3.6}$$

for all $s > 0$, provided we choose φ so that $\varphi(\xi) = 1$ for $|\xi| \le 1/2$. Now $\varepsilon^s = h^{s(1+\mu)}$, and this exponent matches the one in (3.3) provided $\mu = (r-s)/(n+r-s)$. In this case, $s(1+\mu)$ is equal to γ, given in (3.5), and the proposition is proven. □

Next consider a class of functions including those that are piecewise smooth on $\mathbb{T}^n$, with a simple jump across a smooth, embedded $(n-1)$-dimensional surface Σ. We assume the Gauss curvature of Σ is nowhere vanishing. Then such functions u belong to F^r with $r = (n+1)/2$. Assume more generally that

$$u \in F^r \cap \mathrm{PLip}, \qquad r := \frac{n+1}{2}. \tag{3.7}$$

Here $PLip$ denotes the set of functions that are Lipschitz on each component of $\mathbb{T}^n \setminus \Sigma$, with a jump across Σ.

Proposition 3.2 *Let $\Sigma \subset \mathbb{T}^n$ be a smooth $(n-1)$-dimensional surface with nowhere vanishing Gauss curvature. Assume $u \in L^\infty(\mathbb{T}^n)$ satisfies (3.2). Then*

$$\|\rho_h u\|_{L^\infty} \le C h^{2n/(n+1)} \big(\|u\|_{F^r} + \|u\|_{L^\infty} + \|u\|_{PLip}\big). \tag{3.8}$$

Proof. The estimate (3.8) is elementary for $n = 1$, so assume $n > 1$. Then, $r < n$ in (3.7). In this case with $\varepsilon = h^{1+\mu}$, (3.3) yields,

$$\|\rho_h J_\varepsilon u\|_{L^\infty} \;\le\; C h^{(n+1)/2 - \mu(n-1)/2} \|u\|_{F^r}. \tag{3.9}$$

To estimate $\rho_h(u - J_\varepsilon u)$, argue as follows. Choose φ such that $\hat{\varphi}(x)$ is supported on $|x| \le 1$. Then

$$|u(x) - J_\varepsilon u(x)| \le C\varepsilon \|u\|_{\mathrm{PLip}}, \quad x \in \mathbb{T}^n \setminus \Omega_\varepsilon, \tag{3.10}$$

where

$$\Omega_\varepsilon = \{x \in \mathbb{T}^n : \mathrm{dist}(x, \Sigma) \le \varepsilon\}.$$

To estimate the contribution of $u - J_\varepsilon u$ on Ω_ε to $\rho_h(u - J_\varepsilon u)$, set

$$\begin{aligned} N(x, \varepsilon, h, \Sigma) &:= \#\{\lambda \in \Lambda_h : x + \lambda \in \Omega_\varepsilon\}, \\ N(\varepsilon, h, \Sigma) &:= \sup_x N(x, \varepsilon, h, \Sigma), \end{aligned} \tag{3.11}$$

where $\Lambda_h := \{h\ell \in \mathbb{T}^n : \ell \in (\mathbb{Z}/\nu)^n\}$, given $h = 2\pi/\nu$.

When Σ has nowhere vanishing Gauss curvature, it is a standard lattice point estimate (proven using arguments parallel to (3.1)–(3.3)) that

$$N(\varepsilon, h, \Sigma) \le C\varepsilon h^{-n} + C h^{\gamma + 1 - n}, \qquad \gamma = \frac{n-1}{n+1}. \tag{3.12}$$

See, e.g., (3.17) of [12]. When $\varepsilon = h^{1+\mu}$, this gives

$$N(\varepsilon, h, \Sigma) \le C(h^{1+\mu} + h^{1+\gamma}) h^{-n}. \tag{3.13}$$

Equations (3.10) and (3.13) together yield,

$$\|\rho_h(u - J_\varepsilon u)\|_{L^\infty} \le C(h^{1+\mu} + h^{1+\gamma}) \|u\|_{L^\infty} + C h^{1+\mu} \|u\|_{\mathrm{PLip}}. \tag{3.14}$$

Set $\mu = \gamma = (n-1)/(n+1)$. Then the exponent in (3.9) is also equal to $1 + \mu = 2n/(n+1)$, so we have (3.8). $\square$

The simplest example of a function to which Proposition 3.2 applies is $u = \chi_\Omega$, when $\Omega \in \mathbb{T}^n$ is a smoothly bounded region whose boundary $\partial\Omega$ has strictly positive Gauss curvature everywhere. Proposition 3.2 gives

$$\|\rho_h \chi_\Omega\|_{L^\infty} \le Ch^{2n/(n+1)}. \tag{3.15}$$

This is equivalent to well known estimates for lattice point counting problems. Indeed, $\sigma_h \chi_\Omega(0)$ is equal to $(h/2\pi)^n$ times the number of points in $\mathbb{Z}^n$ contained in the dilate $h^{-1}\Omega$. When $n = 2$, for generic such Ω the exponent in (3.15) is known to be sharp; cf. [15]. When $\Omega \subset \mathbb{R}^2$ is a disk, centered at the origin, the exponent is not sharp, and finding the sharp exponent remains a major unsolved problem of analytic number theory. Also for large n and generic smoothly bounded strongly convex regions Ω, identifying the sharp exponent to put on the right side of (3.15) remains an open problem. It is conjectured that the optimal exponent is 2 when $n \ge 5$; cf., e.g., [4].

A number of variants of Proposition 3.2 can be analyzed by similar techniques. We briefly mention one class of examples, including functions that are piecewise smooth and globally Lipschitz on $\mathbb{T}^n$, but whose first order derivatives have a simple jump across Σ. If Σ has nowhere vanishing Gauss curvature, such u belongs to F^r with $r = (n+3)/2$. We will assume more generally that

$$u \in F^r \cap \mathrm{Lip} \cap PC^2, \qquad r = \frac{n+3}{2}, \tag{3.16}$$

where PC^2 means class C^2 on each component of $\mathbb{T}^n \setminus \Sigma$ (extending to the boundary).

Proposition 3.3 *Let $\Sigma \subset \mathbb{T}^n$ be as in Proposition 3.2. Assume $u \in Lip(\mathbb{T}^n)$ satisfies (3.16). If $n > 3$, then*

$$\|\rho_h u\|_{L^\infty} \ \le \ Ch^{4n/(n+1)}\big(\|u\|_{F^r} + \|u\|_{Lip} + \|u\|_{PC^2}\big). \tag{3.17}$$

Proof. This time, in place of (3.9) we have from (3.3)

$$\|\rho_h J_\varepsilon u\|_{L^\infty} \ \le \ Ch^{(n+3)/2-\mu(n-3)/2}\,\|u\|_{F^r}. \tag{3.18}$$

If we choose $\varphi(\xi)$ to be even, we have in place of (3.10)

$$|u(x) - J_\varepsilon u(x)| \le C\varepsilon^2 \|u\|_{PC^2}, \quad x \in \mathbb{T}^n \setminus \Omega_\varepsilon,$$

while

$$|u(x) - J_\varepsilon u(x)| \le C\varepsilon \|u\|_{\mathrm{Lip}}, \quad x \in \Omega_\varepsilon. \tag{3.19}$$

The estimate (3.13) continues to hold, so in place of (3.14) we have

$$\|\rho_h(u - J_\varepsilon u)\|_{L^\infty} \le Ch^{2(1+\mu)}\|u\|_{PC^2} + C(h^{1+\mu} + h^{1+\gamma})h^{1+\mu}\|u\|_{\mathrm{Lip}}. \tag{3.20}$$

Now taking $\mu = \gamma = (n-1)/(n+1)$ in (3.18) and (3.20) yields (3.17). □

Remark. If $n = 1$ or 2, then $r > n$ in (3.15), so by (2.3) we have

$$\|\rho_h u\|_{L^\infty} \le Ch^{(n+3)/2}\|u\|_{F^{(n+3)/2}(\mathbb{T}^n)}, \quad n = 1, 2. \tag{3.21}$$

To treat the analogue of Proposition 3.3 for $n = 3$, we have the following analogue of (3.3) for $u \in F^n(\mathbb{T}^n)$:

$$\|\rho_h J_\varepsilon u\|_{L^\infty} \le Ch^n\Big(\log\frac{h}{\varepsilon}\Big)\|u\|_{F^n} \le C'\Big(h^n\,\log\frac{1}{h}\Big)\|u\|_{F^n}, \tag{3.22}$$

with $\varepsilon = h^{1+\mu}, \mu > 0$, for h small. This replaces (3.18) for $n = 3$, while (3.20) continues to hold, with $\mu = \gamma = (n-1)/(n+1)$. We have:

Proposition 3.4 *In the setting of Proposition 3.3, if $n = 3$, then, for $h < 1/2$,*

$$\|\rho_h u\|_{L^\infty} \le Ch^3\Big[\Big(\log\frac{1}{h}\Big)\|u\|_{F^3} + \|u\|_{Lip} + \|u\|_{PC^2}\Big]. \tag{3.23}$$

One can extend the analysis in Propositions 3.2–3.4 to allow the Gauss curvature of Σ to vanish simply on a hypersurface of Σ, and change sign, producing appropriately modified estimates. One can also consider other classes of surfaces Σ. We will not pursue the details.

4 Random evaluations

The results above estimate the error $\sigma_h u(x) - Mu$, whatever the choice of x. If x is chosen randomly, there is often a good chance the error is smaller. Seeing this simply involves estimating $\|\rho_h u\|_{L^p}$ for some $p < \infty$. We record a few such estimates. Formulas (1.8)–(1.9) yield,

$$\|\rho_h u\|^2_{L^2} = \sum_{j\neq 0}|\hat{u}(\nu j)|^2, \qquad \|\rho_h u\|_{L^p} \le \|(I - \Psi(hD))u\|_{L^p}. \tag{4.1}$$

Hence, parallel to Proposition 2.1, we have

$$\|\rho_h u\|_{L^p} \le Ch^r\|u\|_{B^r_{p,\infty}}, \quad 1 \le p < \infty,\ r > 0, \tag{4.2}$$

using the Besov space $B^r_{p,\infty}$, and

$$\|\rho_h u\|_{L^2} \le C h^r \|u\|_{F^r}, \qquad r > \frac{n}{2}. \tag{4.3}$$

As with (2.2)–(2.3), (4.3) is stronger than (4.2) (at least for $p \le 2$) when $r > n/2$. Note that (4.3) applies to the functions considered in Propositions 3.1–3.3.

Estimate (4.2) is weaker than (2.7) when $rp > n$. We can improve (4.2) by the same sort of interpolation argument used in Proposition 2.2. This yields the following L^2 estimate.

Proposition 4.1 *Assume*

$$1 \le p \le 2, \qquad r > n\Bigl(\frac{1}{p} - \frac{1}{2}\Bigr). \tag{4.4}$$

Then

$$\|\rho_h u\|_{L^2} \le C h^r \|u\|_{B^r_{p,\infty}}. \tag{4.5}$$

Proof. As in (2.8), (4.2)–(4.3) imply that,

$$\begin{aligned} \|\rho_h u\|_{L^2} &\le C h^{r_0} \|u\|_{B^{r_0}_{2,\infty}}, \qquad r_0 > 0, \\ \|\rho_h u\|_{L^2} &\le C h^{r_1} \|u\|_{B^{r_1}_{1,\infty}}, \qquad r_1 > \frac{n}{2}. \end{aligned} \tag{4.6}$$

We get from here to (4.5), via the three-lines lemma, from the corrected version of the (not quite true) complex interpolation identity

$$[B^{r_0}_{2,\infty}, B^{r_1}_{1,\infty}]_\theta = B^r_{p,\infty}, \tag{4.7}$$

with

$$r = (1-\theta) r_0 + \theta r_1, \quad \frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{1} = \frac{1+\theta}{2}, \quad 0 \le \theta \le 1. \tag{4.8}$$

For $B^r_{p,\infty}$ to be achieved in (4.7), given $r_0 > 0,\ r_1 > n/2,\ 0 \le \theta \le 1$, one needs precisely the constraints in (4.4). Details are parallel to those in the proof of Proposition 2.2. □

We get other estimates via a strategy similar to that used for Propositions 3.1–3.3. Parallel to (3.1)–(3.3) we have

$$\begin{aligned} r \in (0, n/2) \quad \Longrightarrow \quad \|\rho_h J_\varepsilon u\|_{L^2} &\le C \varepsilon^r (\nu\varepsilon)^{-n/2} \|u\|_{F^r} \\ &= C' h^{r-(n/2-r)\mu} \|u\|_{F^r}, \end{aligned} \tag{4.9}$$

assuming as usual that $\varepsilon = h^{1+\mu}$. Using this one can obtain an analogue of Proposition 3.1, estimating $\|\rho_h u\|_{L^p}$ when $1 < p \le 2$ and

$$u \in F^r \cap B^s_{p,\infty}, \quad 0 < s < r < \frac{n}{2}. \tag{4.10}$$

We omit the details.

Next compare these estimates with those for the Monte Carlo method. Set

$$\Omega \ := \ \mathbb{T}^n \times \mathbb{T}^n \times \mathbb{T}^n \times \cdots , \tag{4.11}$$

with the product probability measure. Given $p = (x_1, x_2, x_3, \dots) \in \Omega$, define

$$M_K u(p) \ := \ K^{-1}(u(x_1) + \cdots + u(x_K)). \tag{4.12}$$

Then

$$M_K u - Mu \ = \ \frac{(u(x_1) - Mu) + (u(x_2) - Mu) \ + \ \cdots \ + \ (u(x_K) - Mu)}{K}.$$

For $u \in L^2(\mathbb{T}^n)$, the functions $v_j(p) = u(x_j) - Mu$ are mutually orthogonal in $L^2(\Omega)$ so,

$$\|M_K u - Mu\|^2_{L^2(\Omega)} = K^{-2} \sum_{j=1}^{K} \|u - Mu\|^2_{L^2(\mathbb{T}^n)} = K^{-1} \|u - Mu\|^2_{L^2(\mathbb{T}^n)}. \tag{4.13}$$

Therefore,

$$\|M_K u - Mu\|_{L^2(\Omega)} \ = \ K^{-1/2} \|u - Mu\|_{L^2(\mathbb{T}^n)}. \tag{4.14}$$

Note that evaluating $\sigma_h u(x)$ at a given $x \in \mathbb{T}^n$ involves ν^n function evaluations (with $\nu = 2\pi/h$), so we have $K \leftrightarrow h^{-n}$ and $K^{-1/2} \leftrightarrow h^{n/2}$. Thus, when (4.3) applies, evaluating $\sigma_h u(x)$ at a random point $x \in \mathbb{T}^n$ gives, with high probability, a more accurate approximation to Mu than evaluating $M_K u(p)$ at a random point $p \in \Omega$. However, for rougher functions, the Monte Carlo method can be expected to work better.

Let us illustrate the application of (4.2)–(4.3) to $u = v_a$, given for $a \in (0, 1)$ by

$$v_a(x) = x^{-a} f(x) \ \text{ on } \ [0, 2\pi), \tag{4.15}$$

periodically continued to define $v_a \in L^1(\mathbb{T}^1)$, as long as $f \in L^\infty$. For such v_a, we have

$$\begin{aligned} &v_a \in F^a, \quad \text{provided} \ \ f \in F^r, \ r > 1, \\ &v_a \in B^{1/p-a}_{p,\infty}, \quad \text{provided} \ \ 0 < a < \frac{1}{p}, \ p \in [1, \infty), \ f \in B^r_{p,\infty}, \ r > \frac{1}{p}. \end{aligned} \tag{4.16}$$

In such cases, (4.2) and (4.3) yield, respectively,

$$\|\rho_h v_a\|_{L^p} \le C h^{1/p-a}, \qquad 0 < a < \frac{1}{p}, \quad p \in [1, \infty), \tag{4.17}$$

and

$$\|\rho_h v_a\|_{L^2} \le C h^{1-a}, \qquad 0 < a < \frac{1}{2}. \tag{4.18}$$

These results break down to two cases.

Case A. $1/2 \le a < 1$.
Then only (4.17) applies, with $p \in [1, 1/a)$.

Case B. $0 < a < 1/2$.
Then (4.18) applies, and also (4.17) applies, for $p < 1/a$; note that $1/a > 2$. We can interpolate between (4.18) and (4.17), and thereby improve (4.17). Namely, given $q \in (1/2, 1/a)$, $\theta \in (0,1)$, as long as f satisfies both sets of conditions in (4.16),

$$\begin{aligned} \frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{q} \Longrightarrow \|\rho_h v_a\|_{L^p} & \\ &\le C h^{\theta(1-a)+(1-\theta)(1/q-a)} \\ &= C h^{\theta+(1-\theta)/q-a} \\ &= C h^{\theta/2+1/p-a}, \end{aligned} \tag{4.19}$$

so (4.17) is improved by a factor of h^θ. Note that if $p \in (2, 1/a)$, we can arrange that (4.19) applies for some $q \in (p, 1/a)$ for all θ satisfying

$$0 < \theta < \frac{1/p - a}{1/2 - a}. \tag{4.20}$$

To compare the estimates (4.18)–(4.19) with (4.14), note that in Case B, (4.18) is better than (4.14) (with $K^{-1} \approx h$), while in Case A, (4.14) is not applicable.

5 Little o estimates

We examine cases where estimates $\|\rho_h u\|_{L^\infty} \le C h^r$, i.e., $\|\rho_h u\|_{L^\infty} = O(h^r)$, can be improved to $\|\rho_h u\|_{L^\infty} = o(h^r)$, i.e., $\lim_{h\to 0} h^{-r}\|\rho_h u\|_{L^\infty} = 0$. These phenomena are related to the Riemann-Lebesgue lemma. The basic result is the following.

Definition. *If X is a Banach space of functions on $\mathbb{T}^n$, containing $C^\infty(\mathbb{T}^n)$, $\overset{\circ}{X}$ denotes the closure of $C^\infty(\mathbb{T}^n)$ in X.*

Proposition 5.1 *Fix $r > 0$. Let X^r be a Banach space of functions on $\mathbb{T}^n$, containing $C^\infty(\mathbb{T}^n)$, for which one has*

$$\|\rho_h u\|_{L^\infty} \le Ch^r \|u\|_{X^r}, \quad \forall\, u \in X^r. \tag{5.1}$$

Then

$$u \in \overset{\circ}{X}{}^r \implies \|\rho_h u\|_{L^\infty} = o(h^r). \tag{5.2}$$

Proof. Given $\varepsilon > 0$, take $v \in C^\infty(\mathbb{T}^n)$ such that $\|u - v\|_{X^r} \le \varepsilon$. Then

$$\begin{aligned} \|\rho_h u\|_{L^\infty} &\le \|\rho_h(u-v)\|_{L^\infty} + \|\rho_h v\|_{L^\infty} \\ &\le C\varepsilon h^r + C_\varepsilon h^{r+1}. \end{aligned} \tag{5.3}$$

This implies $\limsup_{h\to 0} h^{-r}\|\rho_h u\|_{L^\infty} \le C\varepsilon$, which gives (5.2). □

Recalling results of §2, we deduce that

$$r > n,\ u \in \overset{\circ}{F}{}^r \implies \|\rho_h u\|_{L^\infty} = o(h^r), \tag{5.4}$$

and, given $p \in (1, \infty)$,

$$rp > n,\ u \in \overset{\circ}{B}{}^r_{p,\infty} \implies \|\rho_h u\|_{L^\infty} = o(h^r). \tag{5.5}$$

In the context of Corollary 2.3, since $C^\infty(\mathbb{T}^n)$ is dense in $H^{r,p}(\mathbb{T}^n)$ for $p \in (1, \infty)$, we have

$$p \in (1,\infty),\ rp > n,\ u \in H^{r,p}(\mathbb{T}^n) \implies \|\rho_h u\|_{L^\infty} = o(h^r). \tag{5.6}$$

Also, since $C^\infty(\mathbb{T}^n)$ is dense in $L^1(\mathbb{T}^n)$,

$$r > n,\ (-\Delta)^{r/2} u \in L^1(\mathbb{T}^n) \implies \|\rho_h u\|_{L^\infty} = o(h^r). \tag{5.7}$$

This result, which is closest in spirit to the Riemann-Lebesgue lemma, follows from (5.4). Note incidentally that

$$u \in \overset{\circ}{F}{}^r \iff |\hat{u}(j)| = o(|j|^{-r}), \quad \text{as } |j| \to \infty. \tag{5.8}$$

In the case $n = 1$ and $r = 2$, (5.7) is contained in (1.24) of [2].

6 Trapezoidal rule and midpoint rule estimates

If f is a continuous function on $I = [0, \pi]$, the trapezoidal rule gives the formula

$$\mathcal{T}_h f = \frac{1}{\mu} \sum_{\ell=0}^{\mu-1} \frac{f(h\ell) + f(h\ell + h)}{2}, \tag{6.1}$$

for $h = \pi/\mu,\ \mu \in \mathbb{Z}^+$, as an approximation to

$$Mf = \frac{1}{\pi} \int_0^\pi f(x)\, dx. \tag{6.2}$$

We show that the results of §2 yield estimates on the error in this approximation.

One way to produce a function u on $\mathbb{T}^1$ from f would be to set $u = f$ on $[0, \pi]$, 0 on $(-\pi, 0)$, and periodize. However, this function is likely to be singular at the endpoints. One does not recover in this way accurate error estimates for the trapezoidal rule. In this section we show how a reflection argument extends the accurate estimates to the classical setting of integration over an interval.

Define $\mathcal{E} : C(I) \to C(\mathbb{T}^1),\ \mathcal{E} : L^p(I) \to L^p(\mathbb{T}^1)$, etc., as

$$\mathcal{E}f(x) = f(|x|), \quad -\pi \le x \le \pi, \tag{6.3}$$

recalling $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Then,

$$\begin{aligned} u = \mathcal{E}f \quad &\Longrightarrow \quad \mathcal{T}_h f = \sigma_h u(0),\ Mf = Mu \\ &\Longrightarrow \quad \mathcal{T}_h f - Mf = \rho_h u(0). \end{aligned} \tag{6.4}$$

Thus results of §2 yield the following:

Proposition 6.1 *We have*

$$|\mathcal{T}_h f - Mf| \le Ch^r \|\mathcal{E}f\|_{F^r}, \quad r > 1, \tag{6.5}$$

and, for $p \ge 1$,

$$|\mathcal{T}_h f - Mf| \le Ch^r \|\mathcal{E}f\|_{B^r_{p,\infty}}, \quad r > \frac{1}{p}, \tag{6.6}$$

hence, for $p \in (1, \infty)$,

$$|\mathcal{T}_h f - Mf| \le Ch^r \|\mathcal{E}f\|_{H^{r,p}}, \quad r > \frac{1}{p}. \tag{6.7}$$

We get similar results for the midpoint rule, given by

$$\mathcal{M}_h f = \frac{1}{\mu} \sum_{\ell=0}^{\mu-1} f\Big(h\ell + \frac{1}{2}h\Big), \tag{6.8}$$

with $\mu,\ h$ as in (6.1). In this case,

$$\begin{aligned} u = \mathcal{E}f \quad &\Longrightarrow \quad \mathcal{M}_h f = \sigma_h u\Big(\frac{1}{2}h\Big),\ Mf = Mu \\ &\Longrightarrow \quad \mathcal{M}_h f - Mf = \rho_h u\Big(\frac{1}{2}h\Big). \end{aligned} \tag{6.9}$$

Thus estimates parallel to (6.5)–(6.7) hold for $|\mathcal{M}_h f - Mf|$. For brevity, we omit mention of $\mathcal{M}_h f$ in results below.

We next examine when (6.6) applies to all $f \in B^r_{p,\infty}(I)$, which, we recall, consists of the space of restrictions to I of elements of $B^r_{p,\infty}(\mathbb{T}^1)$ (cf. [14], p. 192). Begin with a result on when we can say $\mathcal{E}f \in B^r_{p,\infty}(\mathbb{T}^1)$.

Proposition 6.2 *Given* $p \in (1,\infty)$*,*

$$\mathcal{E} : B^r_{p,\infty}(I) \longrightarrow B^r_{p,\infty}(\mathbb{T}^1), \qquad \textit{for} \ \ \frac{1}{p} < r < 1 + \frac{1}{p}. \tag{6.10}$$

Proof. A key ingredient in the proof is the fact ([14], p. 154) that $M_{\chi_I} g = \chi_I g$ satisfies

$$M_{\chi_I} : B^s_{p,\infty}(\mathbb{T}^1) \longrightarrow B^s_{p,\infty}(\mathbb{T}^1), \qquad \text{for} \quad -1 + \frac{1}{p} < s < \frac{1}{p}\,. \tag{6.11}$$

One implication of this is that, with $\mathcal{O}g = (\operatorname{sgn} x)\,\mathcal{E}g$,

$$\mathcal{O} : B^s_{p,\infty}(I) \longrightarrow B^s_{p,\infty}(\mathbb{T}^1), \quad -1 + \frac{1}{p} < s < \frac{1}{p}\,. \tag{6.12}$$

Now, given $f \in B^r_{p,\infty}(I)$, to check whether $\partial_x \mathcal{E}f \in B^{r-1}_{p,\infty}(\mathbb{T}^1)$, we use

$$\partial_x \mathcal{E} f = \mathcal{O}\partial_x f. \tag{6.13}$$

So (6.10) follows from (6.11)–(6.13). $\square$

Putting together (6.6) and (6.10), we obtain:

Corollary 6.3 *Given* $p \in (1,\infty)$*,*

$$|\mathcal{T}_h f - Mf| \le Ch^r \|f\|_{B^r_{p,\infty}(I)}, \qquad \textit{for} \quad \frac{1}{p} < r < 2. \tag{6.14}$$

Proof. The results (6.6) and (6.10) directly give (6.14) for $r \in (1/p, 1 + 1/p)$. If $r \in [1 + 1/p, 2)$, we can choose $q \in (1, p)$ such that $r \in (1, 1 + 1/q)$ and deduce that $|\mathcal{T}_h f - Mf| \le Ch^r \|f\|_{B^r_{q,\infty}(I)}$, which is dominated by $Ch^r \|f\|_{B^r_{p,\infty}}$. □

Using $H^{r,p}(I) \subset B^r_{p,\infty}(I)$, we have:

Corollary 6.4 *Given $p \in (1, \infty)$,*

$$|\mathcal{T}_h f - Mf| \le Ch^r \|f\|_{H^{r,p}(I)}, \qquad \text{for} \quad \frac{1}{p} < r < 2. \tag{6.15}$$

The endpoint case $r = 2$ of (6.15) is well known and elementary:

$$|\mathcal{T}_h f - Mf| \le Ch^2 \|f'\|_{BV}. \tag{6.16}$$

This can be deduced from (2.21) by a reflection argument, or one can get it directly, without passing to $\mathcal{E}f$ on $\mathbb{T}^1$. See, e.g., [13], p. 128, Exercise 14. Note also that (2.20) plus a reflection argument gives

$$|\mathcal{T}_h f - Mf| \le Ch \|f\|_{BV}, \tag{6.17}$$

given f continuous and of bounded variation on I. For $r \in (1, 2)$, the estimate in (6.15) also follows from (6.16)–(6.17), by an interpolation argument, though this does not work for $r \in (1/p, 1)$. In any case, (6.14) does not follow by interpolation from (6.16)–(6.17).

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