Supplementary Problems for MACS2 spring 2009 **J. RAUCH**

Each student must solve two problems. These can be any problems from the sections in *Hyperbolic Partial Differential Equations and Geometric Optics*, or from among the following additional questions. The two problems should be chosen on the topics of two different sections.

Introductory lecture.

Supp.1. Energy law for the Klein-Gordon equation. i. *For real solutions of the Klein-Gordon equation,*

$$u_{tt} - \Delta u + u = 0,$$

prove the differential energy law

$$\partial_t (u_t^2 + |\nabla_x u|^2 + u^2) \ + \ \mathrm{div}(2\, u_t \nabla_x u) \ = \ 0.$$

ii. *Derive from this the fact that if* $u(0,x) = u_t(0,x) = 0$ *for* $|x| \geq R$*, then* $u = 0$ *for* $|x| \geq R + |t|$*.*
Hint. In order to have integrals over finite sets it is easier to prove that u vanishes on $\{R + |t| \leq |x| \leq M - |t|\}$ with $M > R$ arbitrary.
iii. *For compactly supported initial data show that*

$$\int u_t^2(t,x) + |\mathrm{grad}\, u(t,x)|^2 + u^2(t,x)\ dx$$

is independent of t*.*

Discussion. This shows that the $L^2(\mathbb{R}^d)$ norm of u is bounded uniformly in time. For D'alembert's wave equation the L^2 norm can grow in time as the next examples show. If $f(s)$ is a smooth function on $\mathbb{R}$ which is equal to 1 for $s \leq 0$ and vanishes for s large positive, then

$$u = f(x-t) - f(x+t) \quad \text{for} \quad d = 1, \qquad \text{or} \qquad u = \frac{f(|x|-t) - f(|x|+t)}{|x|} \quad \text{for} \quad d = 3,$$

is a smooth, compactly supported solution of $u_{tt} = \Delta u$ and for t large there is a constant $c > 0$ so that

$$\int u^2(t,x)\ dx \ \geq \ c|t|^d\,.$$

This growth of the $L^2(\mathbb{R}^d)$ norm is in sharp contrast to the Klein-Gordon equation. In physical problems where D'Alembert's equation occurs, either the energy is measured by $\partial_{t,x} u$ (e.g. acoustics) or there are supplementary equations which guarantee that the L^2 growth does not occur (e.g. Maxwell's equations).

iv. Show that if the Cauchy data vanish in $|x| \leq R$ then the solution vanishes in the cone $\{(t,x) \ : \ |x| \leq R - t, t \geq 0\}$.

Supp.2. Plane waves and finite speed. The fundamental finite speed assertion for the wave equation asserts that if $u \in C^2(\mathbb{R}^{1+d})$ satisfies

$$u_{tt} = c^2 \Delta u, \qquad c > 0,$$

and,

$$u(0,x) = u_t(0,x) = 0 \quad \text{for} \quad |x| \leq R,$$

then,

$$u(t,x) = u_t(t,x) = 0, \quad \text{for} \quad |x| \le R - c|t|.$$

Use plane waves or spherical waves to show that for any $\sigma < c$ *one cannot conclude that* $u = 0$ *for* $|x| \le \sigma|t|$. **Discussion.** If one could so conclude it would say that signals traveled no faster than σ which would have been a stronger conclusion. Draw a sketch!

Supp 3. The method of characteristics and the Fourier mehod agree. Show that when $d = 1$ the formula for the solution of the Cauchy Problem by d'Alembert's method yields the same answer as the formula

$$\hat{u}(t,\xi) \;=\; \cos(|\xi|t)\,\hat{f}(\xi) \;+\; \frac{\sin(|\xi|t)}{|\xi|}\,\hat{g}(\xi)\,.$$

§**1.4.**

Supp.4. Group velocities. This problem looks more closely at the direction of motion of the high frequency solutions constructed in §1.4.

i. First consider solutions of the wave equation oscillating rapidly in the direction $\underline{\xi} \ne 0$ rather than the special case $\mathbf{e}_1$ treated in the notes. Consider the solution of the wave equation

$$u = \frac{1}{(2\pi)^{d/2}} \int e^{i(x\xi - |\xi|t)}\;\hat{\gamma}(\xi - \underline{\xi}/\epsilon)\; d\xi.$$

Compute the Cauchy data for this solution. Perform a change of variables as in the notes and then a Taylor expansion to order 1 to derive an approximate solution. Write this approximate solution as an exact plane wave solution times and amplitude which is rigidly translated at a velocity $\mathbf{v} = \mathbf{v}(\underline{\xi})$. Find $\mathbf{v}$.

Next consider the anisotropic wave equation

$$u_{tt} = u_{xx} + 4u_{yy}\,.$$

ii. Show that $f(\tau t + \xi x)$ is a solution for arbitrary f provided that the dispersion relation

$$\tau^2 = \xi_1^2 + 4\xi_2^2, \qquad \xi \in \mathbb{R}^2, \qquad \tau = \tau_\pm(\xi) = \pm\sqrt{\xi_1^2 + 4\xi_2^2}$$

is satisfied.

Consider the exact high frequency solution

$$\frac{1}{\sqrt{2\pi}} \int e^{i(\xi x - \tau_\pm(\xi)t}\;\hat{\gamma}(\xi - \underline{\xi}/\epsilon)\; d\xi\,.$$

iii. Compute the Cauchy data of these highly oscillatory solutions.

iv. Use a Taylor expansion as in the notes to compute an approximate soution whose leading term is a plane wave times an amplitude which is transported at a velocity $\mathbf{v}_\pm(\underline{\xi})$.

Discussion. The velocity is called the **group velocities**. Looking at the derivation with Taylor expansion shows that it is given by

$$-\nabla_\xi \tau_\pm(\underline{\xi}),$$

which is the classic formula in science texts.

v. Show that the group velocities computed above satisfies $\mathbf{v}.\xi = -\tau$ verifying that they satisfy the condition demanded of phase velocities.

Supp.5. Method of Images. i. *Verify that* $u := e^{-|x|}/2$ *satisfies*

$$\Big(- \frac{d^2}{dx^2} + 1 \Big) u \;=\; \delta(x)\,.$$

Hint. Use the definition of distribution derivatives.

ii. *Find a solution of*

$$\Big(- \frac{d^2}{dx^2} + 1 \Big) u \;=\; \delta(x-a)\,.$$

iii. *Suppose that* $a > 0$. *Construct a solution* $u(x)$ *continuous on* $x \geq 0$ *and so that*

$$\Big(- \frac{d^2}{dx^2} + 1 \Big) u \;=\; \delta(x-a) \qquad u(0) = 0\,, \qquad \lim_{x\to\infty} u(x) \;=\; 0\,.$$

Discussion. It is not hard to show using the maximum principal, that there is only one such solution u.

iv. *Suppose that* $a > 0$. *Construct a solution* $u(x)$ *continuous on* $x \geq 0$ *and so that*

$$\Big(- \frac{d^2}{dx^2} + 1 \Big) u \;=\; \delta(x-a) \qquad \frac{du(0)}{dx} = 0\,, \qquad \lim_{x\to\infty} u(x) \;=\; 0\,.$$

Hint. Even functions satisfy the Neumann condition at $x = 0$ while odd functions satisfy the Dirichlet condition.

§**1.5.**

Supp.6. Fish location and Snell's law. *Standing on shore a fisherman with eyes at height* h *above the water looks into the water at a fish which is swimming at depth* d *below a point* L *units of distance away. The line of sight passes* A *units above the fish making the fish look to be less deep than it really is. Compute a formula for* A *in terms of* h, d, L. **Discussion.** A spear fisherman who does not correct for this effect will throw the spear A units above the fish. **Hints.** The formulas require the solution of a nonlinear equation that is not explicitly solvable. One strategy is the following. Denote by n the index of refraction. Express the length L as a sum of two intervals the first being from the fisherman to the point where the line of sight hits the water to find

$$L \;=\; h\,\tan\theta_i \;+\; d\,\tan\theta_r\,.$$

Write this in terms of $\sin\theta_i$ and $\sin\theta_r$. Use Snell to eliminate $\sin\theta_r$. Conclude that

$$\sin\theta_i \;=\; g^{-1}(L)\,,$$

where,

$$g(x) \;:=\; h\,k(x) \;+\; d\,k(x/n), \qquad k(x) \;:=\; \frac{x}{\sqrt{1-x^2}},$$

which is an strictly increasing invertible map $]0,1[\mapsto]0,\infty[$. If you choose a different attack the nonlinear equation to solve may be different. I don't know if there is a simpler formula. Once $\sin\theta_i$ is known one can compute A.